

THE FOUNDATIONS OF MATHEMATICS

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The following article presents the results of an investigation on various levels into the nature and foundations of mathematics. The basic level I may call the methodological level, the precise nature of which will be determined more fully as we proceed. Other levels involved are that of mathematics proper, that of metamathematics¹—where this is not restricted to finitary methods, the pedagogical level, and the level of scientific applicability. The presentation will be in a somewhat popular nontechnical form, and this for two reasons. First, specialization has separated the levels in question, and a presentation on any one of them would be meaningful only to those familiar with that particular viewpoint. Second, researches on any but the basic level already mentioned have failed to yield genuine clarity; and since this methodological level has a touch of novelty about it, familiarity with it can neither be presupposed nor generated here.

Now, a successful clarification should meet squarely six major requirements. First, it must account for the historical development of mathematics. So it must face up, for example, to the transition from prime numbers to polynomial ideals, the extension of the notion of parallelism and of metric from Euclid to Riemann and beyond, the developments in integration theory, in topology, and in lattice theory.² Second, it must account for the process of evolution of mathematics in the individual mind, as experienced and described by pedagogues and psychologists. Third, it must account for the happy interplay of the experimental sciences with mathematics. Fourth, the successful

clarification must account for the various other views on the same subject. Fifth, it must say just enough, not so much as to appear to solve genuine mathematical questions, not so little as to leave mathematics without a future. The significance of this requirement will appear in the conclusion.

Sixth, the clarification must square with the personal experience of the individual mathematician, and I place this demand last not because it is least but because it is the basis from which clarification springs. No doubt the notion that one might clarify the foundations of mathematics by introspection is distasteful to many others besides Gottlob Frege.³ However, the introspection in question is not the barren or helpless looking into oneself popular with some Scholastics and many existentialists. It is rather the process of catching oneself in the act of doing both mathematics and metamathematics. It goes beyond Hadamard's effort in his little book,⁴ yet it is not unrelated to it. In this connection I quote the following comment on Hadamard's reflections on the working of mathematicians' minds:

Such things may strike us strange and rather fascinating, a strand of queerness enlivening the dull desert of scientific thought, arid stretches of logic. We may dismiss them lightly and pass on to the serious consideration of what thought and understanding are in terms of the words that philosophers have been accustomed to use. But we may be quite wrong in this. We may miss the turning leading to an understanding of understanding.⁵

It is precisely this turning leading to an understanding of understanding that I have taken; and before I go on to discuss the results I should like to remark that the understanding of understanding in

³Metamathematics may be said to have originated with David Hilbert's efforts to prove the consistency of classical mathematics by first expressing it in axiomatic form, making this formal system the object of a proof theory or metamathematics. This theory was to use only intuitively convincing methods, called by Hilbert "finitary methods." As the theory advanced, finitary methods were seen to be inadequate.

²For a general account of these

subjects and their development, cf. E. T. Bell, *The Development of Mathematics* (New York: McGraw-Hill, 1945).

⁴*The Foundations of Arithmetic*, trans. J. L. Austin (Oxford: Basil Blackwell, 1953), p. vi.

⁵Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* (Princeton: Princeton Univ. Press, 1945).

⁶J. L. Synge, *Science, Sense and Non-sense* (London: Cape, 1951), p. 112.

question is reached only insofar as one moves through personal acts of understanding to an appreciation of one's own experience of understanding. For this reason what follows may on mere reading ring hollow and not true. If, however, it is to be judged fairly it must be judged not by comparison with other theories but by comparing it with one's own personal experience of mathematics.

Generally, when the nonscientist asks me what understanding is, I try to give the experience of understanding by some simple geometry. With mathematicians such a method is not so sure to succeed, for the simple problem in geometry is usually no problem at all—the solution is too obvious. However, I will take here one simple example, the significance of which will not be missed, and I will make some comments on the processes it involves.

In a circle of, say, unit radius, we draw two perpendicular diameters. Taking any point P on the circumference, we drop perpendiculars PR and PS on the two diameters. Joining R to S , I ask my nonscientific friend (or in the present case the reader), What is the ratio of RS and the radius? At this stage my friend looks puzzled and perhaps tries calculation. Eventually I draw an extra line. I simply join P to the centre, and my friend utters his own version of Archimedes' "Eureka!" Now, while the element of surprise is absent for the geometer, a few interesting remarks may be made on the process. First, the act of understanding or insight involved in the solution was dependent on the diagram, and indeed even on the modification of the diagram for the nongeometer. Second, what was grasped in the insight was a relation, the relation between RS and the radius. Third, that grasp can be formulated or thrown into syllogistic form—and here some light is thrown on a feature of Aristotelian logic often misrepresented. The question raised was one concerning the relation of RS to the radius, OM , say. The question indeed was one of finding a middle term, and the middle term was supplied as soon as one adverted to the significance of OP . Only then is the syllogism constructed. To coin an expression for this, let us say that the insight is crystallized into a syllogism. The points raised in this simple example will recur later, and their importance will become evident. While on the topic of

crystallizing insights, however, let me give two examples of insights crystallized not into syllogisms but into axioms.

The first example is a casual insight which occurs regularly in Euclid, the insight that a line which contains a point of one side of a triangle must contain a point of one of the other sides. The insight was formulated as an axiom of order by Pasch (1890), and its effect is to liberate us to some extent from diagram.

The second example is an assumption occurring in Cantor's work,⁶ which was first formulated by Zermelo (1904), the famous axiom of choice.⁷ This axiom is concerned with the possibility of selecting a definite representative element from each nonvoid subset of a given set.

Now, what I illustrated by simple example can happen on a larger scale, and then what is formulated is not just a syllogism or an axiom but, for example, the whole of Euclidean geometry. Further, insofar as one eliminates casual insights and merely nominal definitions such as are present in Euclid, one achieves the ideal of proper axiomatization aimed at by Peano and his followers. If I might venture a definition, I should say that an ideal axiom system is a related set of terms and relations, in which the relations determine the terms and the terms the relations. This definition may be seen to include Hilbert's notion of implicit definition. Yet it does more, for it lays emphasis on the fact that the terms are defined precisely by the relations and vice versa; and in doing so it excludes the notion of what might be called "absolute definitions," a notion that has had such an adverse effect on both philosophy and science in past centuries. The false notion is both present and partially rejected by Pasch in the following remark:

If geometry is to be deductive, the deduction must everywhere be independent of the meaning of geometrical concepts, just as it must be independent of diagrams; only the relations specified in the propositions and definitions employed may legitimately be taken into account.⁸

Pasch rightly laid emphasis on the significance of the relations, but he

⁶Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers* (New York: Dover Pubns.), pp. 105, 161, 204.

⁷Cf. E. W. Beth, *The Foundations of Mathematics* (Amsterdam: North-Holland Pub. Co., 1959), p. 376. I shall use

this as a standard work of reference throughout, denoting it as "Beth."

⁸M. Pasch, *Vorlesungen über neuere Geometrie* (Leipzig: Teubner, 1882), pp. 98-99.

⁹G. Birkhoff, *Lattice Theory* (New York, 1948).

was a child of European philosophy in not identifying the meaning of the geometrical concepts with the relations. The most important example of such oversight and confusion concerns "quantity." On the present view quantity is anything that can serve as a term in a numerical ratio; and inversely a proportion is a numerically definable ratio between quantities. Quantities and proportions are terms and relations such that the terms fix the relations and the relations fix the terms.

Modern mathematics is rich in examples of axiom systems which tend towards the above idea. As a very powerful instance one might mention the axiomatic presentation of lattice theory,⁹ in which the terms are not, as some authors would have it, meaningless but are precisely defined by the relations.

While it would be logical to discuss at this stage the analytic nature of basic propositions, the manner of generating axiom systems, and the process of selecting relevant ones, such a discussion would take us too far afield. I cannot, however, omit a brief treatment of the nature of the deductive expansion by which one passes from the basic axioms to the theorems in any particular branch. I cannot agree with the common view that this process is a mere logical expansion of conceptual premises. Let me illustrate the point with a simple and obviously imperfect axiom system. While I use the words "point," "line," and so on, they are not to be taken at their face value.

Axiom 1. Every line is a collection of points.

Axiom 2. There exists at least two points.

Axiom 3. If p and q are points, then there exists one and only one line containing p and q .

Axiom 4. If L is a line, then there exists a point not on L .

Axiom 5. If L is a line and p is a point not on L , then there exists one and only one line containing p that has no point in common with L .

One reason why I use this axiom system is that it can have a real model which will serve as an illustration later. One need only add a sixth axiom restricting the number of points to four, and then the

real model is provided by four eccentric old gentlemen who form six clubs, two men in each club. Axiom five for the model then states that there exists one and only one club containing the gentleman p which contains no member of a club not containing the gentleman p . However, our immediate concern is the deduction of Theorem A , "Every p is on at least two L s." We consider two lines to be different when they are different collections of points.

The proof is more or less obvious according to one's mathematical ability. Thus if p is any point, we have a second point q by Axiom 2. Axiom 3 gives us a containing line, say L' , for p and q ; and Axiom 4 a further point r not on L' . By Axiom 3 there exists a line L'' containing p and r ; and since L' does not contain r , we conclude that L' and L'' are different collections of points and so different lines.

It is to be noted first that the theorem is not proved without symbols. Second, the proof involves a series of insights into the relations of terms, relations, and axioms. Third, these insights can be crystallized, all assumptions made explicit, and the whole cast into deductive form. Lastly, the proof is understood properly only when it is grasped as a whole and when it can be explained intelligently and not just repeated mechanically.

In what we have so far discussed of mathematics, one basic type of question has continually recurred, the type of question which I call the "what" question. So, for example, we had the questions, "What is the relation between the line RS and the radius?" "What relations hold between the axioms?" and so on. The "what" question is a question for direct understanding, and the answer is some form of definition or relation.

There is, however, a second fundamental type of question which I call the "is" question; for example, "Is it true?", "Is it an axiom?" "Is it consistent?" The proper answer to this type of question is yes or no, a judgment. Furthermore, the answer, to be of value, must be an intelligent one; and so it too must spring from understanding, an understanding which may be called reflective to distinguish it from the direct understanding of the "what" question. Now, in mathematics, while judgments undoubtedly do occur, still the stress is on the "what" questions. On the other hand, in metamathematics, while there is an abundance of theory, the stress is on the "is" questions. So there

are the three basic metamathematical questions regarding any axiom system:

(a) Are the axioms independent, or is one axiom derivable from the others?

(b) Is the system consistent? If I persevere long will I arrive at a contradiction, P and not- P ?

(c) Is the system complete; that is, does the system enable me to prove one out of each two contradictory statements, R and not- R , legitimately expressed in the terminology of the system? "Legitimately" here means according to rules for the formation of formulae, rules, for example, which govern the distribution of parentheses.

Before further discussion it will be helpful to note that we have so far distinguished five basic components of cognitional structure which I may designate as experience (on the sensible level diagram, and so on), direct understanding, formulation, reflective understanding, and judgment.

Judgment—or more precisely the reflective understanding leading to judgment—can be centrally involved with one or other of the components. Thus one may ask, "Am I seeing, hearing, imagining, this or that?" and then one's concern is with the first component. One may ask, "Have I understood properly?" and then it is direct understanding that is being scrutinized. Third, one may ask, "Does my theory hold together?" This is the type of question central to metamathematics. It is centred on formulation; and if one visualizes the theory cast into deductive form, then it is scanned from top to bottom by the questions (a), (b), (c), mentioned already. So one examines axioms, deductive processes, and the extent of the theory. This, of course, is simplifying the situation somewhat, since the three basic questions are in fact interrelated. Fourth, one may ask, "Is my theory true?" This is the question which occurs primarily in science; it is answered in the affirmative only insofar as a given theory is verified.

Let us return to the question of consistency which is obviously the most pressing. There are three main approaches to the problem.

The first approach is to search for an actual model. If one is found, then one has verified the theory, and one concludes from the existence of the real model that the theory must be consistent. So, for the simple axiom system which we discussed earlier, I pointed out that there could be a real model insofar as any four people might form the required six clubs. This method is clearly related to the fourth type of judgment mentioned above.

The second method is to produce what I call a semi-imaginary model. Examples are the models of Poincaré and Beltrami for hyperbolic geometry, these two models being neatly brought together by Klein as projections of a sphere on different planes.¹⁰ I call these semi-imaginary, since, while they make use of an imagined model, they refer back to a second theory—in the examples to Euclidean geometry. One might consider the stress in this method to be on the first and second types of judgment discussed above, though none of these distinctions is rigid. This method, moreover, yields only relative consistency.

Third, one can tackle the problem of consistency more or less according to the Hilbert program.¹¹ This last method is closely connected with the third type of judgment discussed above. One is heading for success here insofar as one generates an ideal axiom system, grasps the axioms as analytic, and makes explicit the deductive procedures allowed, so that one has ensured that all casual insights have been crystallized. By doing this one is casting the theory into a form in which one can grasp the evidence for judgment on its consistency. One may even formalize one's grasp of the evidence, and then one has a formal metasystem. So, for example, one formulates a consistency proof for propositional logic by using a mapping onto a domain of two objects. Again, Gödel's first incompleteness theorem may be described as demonstrating that, in a system broad enough to contain all the formulae of a formalized elementary number theory, there exist theorems that can neither be proved nor disproved within the system. The manner in which he arrived at his theorem involved a formaliza-

¹⁰H. S. M. Coxeter, *Introduction to Geometry*, (New York: Wiley & Sons, 1961), p. 289.

¹¹Cf. n. 1.

¹²Beth, p. 593. For a nontechnical presentation, see E. Nagel and J. R. Newman, *Gödel's Proof*, in *The World of*

Mathematics, ed. J. R. Newman (London: Allen & Unwin, 1960), III, 1668-95.

¹³A general account of these three main approaches and their development is given in R. A. Wilder, *Introduction to the Foundations of Mathematics* (New York: Wiley & Sons, 1952).

tion of the metasytem within the arithmetic. This was done essentially by a judicious use of prime numbers which gave to each formula a unique number, called its Gödel number, and to relations in the metasytem definite relations between Gödel numbers. I cannot go into Gödel's work further here, but I wish to relate his second theorem to the present methodology and thus also highlight a definite limitation of the Hilbert program.¹²

Gödel succeeded in producing a formula of the arithmetic which, when interpreted in the metasytem, meant "A is consistent," A being the arithmetic. He then showed that if A is consistent, then the formula corresponding to "A is consistent" cannot be proved in A. The proof program thus receives a setback in that a consistency proof of a given system will presuppose a stronger system than the one under examination.

Consider now the Hilbert program from the methodological point of view. From that point of view what is required is a formulated judgment falling on the formulated theory, A. The evidence for this judgment lies in a grasp of the analytic nature of the axioms, of the reliability of the allowed deductive processes, and so on. The problem of systematically formulating a consistency proof is that of formulating the grasped evidence for consistency. Grasping the theory A is only a part of this evidence, and so we cannot expect a full formulation of the evidence within A. In making this methodological comment I am not of course implying that it is independent of the work of Gödel. The methodology and the metamathematics, or mathematics, should indeed always move forward together in a complementary fashion. To this I will return in the conclusion.

Having given some account, by means of a schematic presentation of cognitional structure, of the general movement in both mathematics and metamathematics, I would like to discuss briefly a few of the other schools of thought in terms of that account. Although there is a large range of opinions, both Scholastic and non-Scholastic, I restrict myself here to three of the modern tendencies: logicism, intuitionism, and formalism.¹³

Logicism, roughly, would have mathematics cast into a *logica magna* in which one can pass by deduction to all the theorems of mathe-

matics.¹⁴ Clearly the stress in logicism is on the third component in our schema, on formulation or fully axiomatized mathematics. Its failure, which could be traced historically, lies in not recognizing the role of insight in formulation, in considering deduction to be merely a conceptual, even tautological, expansion, and in not sufficiently acknowledging the openness of mathematics. Known mathematics at a given stage may well be thrown onto a *logica magna*, where deduction is understood correctly. But the process would demand, as remarked earlier, the "crystallization" of all "casual" insights; and unless mathematicians are silenced, the latter will always run ahead of the former.

Next, a few remarks on Brouwer's intuitionism.¹⁵ It is interesting to note that the maxims of the intuitionists re-echo to some extent our own methodological principles. For example, intuitionists would claim that it is not possible to penetrate the foundations of mathematics without paying due attention to the conditions under which the mental activity proper to mathematicians takes place. The program was not followed up successfully, however; instead, the school has developed its own version of mathematics. Intuitionism lays stress, for example, on the need for constructive proofs, on the inadequacy of the principle of the excluded middle, and on the notion of absurdity as basic in mathematics. These stresses spring from the fact that the intuitionists' attention is on the insight prior to formulation, its incompleteness and its presuppositions. This is borne out, for example, by considering the manner in which the principle of the excluded middle is limited on this level. On the level of judgment the principle of the excluded middle enjoys definite validity; if a judgment occurs it must be either an affirmation or a denial. On the level of direct understanding, how-

¹⁴Logicism may be traced to Gottlob Frege, who, in his *Grundlagen der Arithmetik* (Breslau, 1884), gave a summary reduction of arithmetic to logic. His work, however, was not widely known before Bertrand Russell arrived at some of his conclusions independently. The latter advanced the program considerably.

¹⁵Although L. E. J. Brouwer is considered to be the founder of intuitionism, he was preceded by L. Kronecker, who insisted on the notion of mathematics as a construction on the basis of "intuitively

given" natural numbers. Kronecker is popularly remembered by his after-dinner-speech remark, "The integers were made by God, but everything else is the work of man."

¹⁶The evidence for the thesis of Alonso Church, which may be considered as a generalization of that of Gödel, is fully discussed in S. C. Kleene, *Introduction to Metamathematics* (Amsterdam: North-Holland Pub. Co., 1952), pp. 298-386.

¹⁷For a discussion of the Skölem-Iöwenheim paradox, cf. Beth, pp. 488-90, 513-16.

ever, there are not two but three alternatives with regard to any formulated proposition; for not only can one accept or reject, but one can also go on to seek a better understanding and so a more adequate formulation.

Hilbert and his proof program have already been favorably mentioned in relation to the ideals of axiomatization, of implicit definition, and of casting mathematical theories into a form suitable for some judgment on consistency. Needless to say, we could not enter into any of the details of the actual achievements of the program or its modifications. The fact that theorems like those of Gödel and Church¹⁶ put limits to the program does not deprive the method of its value as contributory to the understanding of mathematics. Weakness on the nature of deduction and on the meaningfulness of terms betrayed by this as by other approaches are points which have already been discussed.

I add some brief methodological comments on the various "paradoxes." These I divide into five groups in order of ascending complexity. I will, however, omit the fifth group, which includes paradoxes springing from metamathematics such as the Skölem-Löwenheim model paradox, since their discussion would be too technical.¹⁷

The first group may be classed as paradoxes of denotation. For example, consider the inference:

343 contains 3 figures,

$$343 = 7^3,$$

therefore 7^3 contains 3 figures.

Here, as in the case of many of the paradoxes, there are various solutions formulated by different authors. These solutions, I would claim, are correct insofar as they crystallize the casual insight which provides the solution on the methodological level. On this level the casual insight consists in grasping the distinction between properties which pertain to numbers on the experiential level and properties which pertain to them insofar as they are understood. Furthermore, the solution is adequate, in this as in other paradoxes, insofar as it excludes

by means of axioms and notation the reoccurrence of similar paradoxes, removing thus the burden from the casual insight to the symbolism.

The second group may be classed as dictionary paradoxes, and I will take as example the Berry paradox. Consider the finite set P of sentences which contain at most fifty words from a given dictionary. Consider further the subset Q of these which define a natural number. Since the set Q is finite, there are natural numbers not defined in Q . The first of these, taking the numbers in their natural order, we call the Berry number. Now consider the sentence:

The Berry number is the first number, in accordance with the usual arrangement of natural numbers, which cannot be defined by means of a sentence containing at most fifty words, all of them taken from our dictionary.

This sentence contains only thirty-seven words, but it defines the Berry number. So the Berry number is defined in Q' .

Again, while elaborate solutions can be presented, to be correct they must take account of a basic distinction which is as important as it is apparently trivial. It is the distinction between description and definition or explanation. The thirty-seven-word statement does not in fact define the Berry number; it merely describes it. To bring out the importance of this distinction in other fields, it is worth noting that one can describe electrons as particles or waves; but if one wishes to define or explain them—which is what the physicist seeks to do—one must have recourse to mathematically formulated and verified equations.

The third group of paradoxes includes what are called semantic paradoxes. The simplest example is the "liar paradox." Somebody makes the statement, "I am a liar." Is the statement true or false? If it is true, then he is a liar; and so it is false. If it is false, then he is not a liar; and so it is true.

Tarski's discussion of this paradox does not seem to be adequate, nor, as far as I know, has a clearly formulated systematic solution

¹⁸*Ibid.*, pp. 335-45.

¹⁹Bertrand Russell, *The Principles of Mathematics* (2nd ed.; London: Allen & Unwin, 1935), chap. X.

²⁰Beth, pp. 381-408.

²¹Bärbel Inhelder and Jean Piaget, *The Growth of Logical Thinking from*

Childhood to Adolescence (New York: Basic Books, 1958). Also the many independent works of Piaget such as *The Child's Conception of Number* (London: Routledge & Kegan Paul, 1952).

²²Cf. n. 2.

appeared.¹⁸ Methodologically the basis of solution is as follows. First, the statement "I am a liar" can be written down, represented on the sensible level; and then, while it has meaning for the reader, it is still merely so many black marks ordered against a white background. Again the reader may think the statement "I am a liar"; he can merely consider it, as he is doing now, without judging. But he cannot go on to make it a judgment, for judgments proceed from intelligent grasp of evidence; and evidence for the present proposition is lacking unless one has actually lied, in which case the correct judgment is "I have lied." However, one can also utter aloud the sounds "I am a liar," but then these sounds are on a level equivalent to that of print on paper.

The fourth group of paradoxes consists of the paradoxes of set theory. The most familiar example is perhaps that of Russell: Is the set of sets which are not members of themselves a member of itself or not?¹⁹ Here again I restrict myself to a methodological comment.

There are two ways of "defining" a definite set, either by identifying the members (real or imagined) individually or by defining the set intelligently. The first method presents no basic difficulty. As regards the second method, however, paradoxes may emerge if in fact particular sets are not intelligently defined. The problem is to crystallize, or axiomatize, the insight by which one grasps this, so as to exclude systematically further occurrences. Various solutions have emerged, the most familiar perhaps being that of Zermelo, at least in one of its modified forms.²⁰ In each of these some restriction is imposed on the type of class that can be condensed into a set. The present state of the discussion of the notion of set in general, however, is not a very happy one. Methodologically speaking, I should say that some obscurity would be removed if more emphasis were laid on the notion that the set and its members are relation and terms in which the relation fixes the terms and the terms fix the relation.

My account has been necessarily sketchy, and if I claim that the solution presented meets all six requirements listed at the beginning, I must do so without justifying that claim here. That justification would indeed entail a systematic discussion; for example, the findings of a historian such as E. T. Bell,²¹ of a psychologist such as J. Piaget,²²

of a mathematician such as J. Hadamard.²³ Sufficient indications have been given, however, to show that the claim is not groundless. I will conclude with a word about the background of this work, adding references to enable the interested reader to complement what has been here discussed, and some remarks on the broader significance of the method here used.

The fundamental element in the solution presented is of course the methodology which I have all too briefly described. For this methodology I am indebted to the works of Bernard Lonergan, s.j., especially to his book *Insight*,²⁴ and to his articles "The Concept of *Verbum* in the Writings of St. Thomas Aquinas."²⁵ Many points which I should have discussed here have in fact been omitted because they are adequately treated in these works. Such points are the object, nature, and heuristic definition of mathematics,²⁶ the nature of relations,²⁷ the genesis of basic propositions and their analytic nature,²⁸ the nature of probability,²⁹ the process of mounting generalization,³⁰ and the interplay of mathematics with science.³¹

Lastly, a few remarks on the broader significance of the present approach. Three levels have been successfully distinguished: mathematics proper, metamathematics—in which I would like to include also a substantial section of logic—and methodology of mathematics. The distinction between mathematics and metamathematics is not strict; the domain of methodology is, however, more clearly defined. This methodology is such that it gives expression to something which (a) is basically the same in, for example, Euclid, Eisenhart, and Einstein, (b) can be more fully formulated as mathematics advances, (c) is scientific, since its scientific formulation is constantly checkable in the changing data of cognitional fact.

I would contend that this methodology is identifiable with the philos-

²³Cf. n. 4.

²⁴B. J. Lonergan, s.j., *Insight: A Study of Human Understanding* (London: Longmans, Green & Co., 1957).

²⁵*Theological Studies*, VII (1946), 349-92; VIII (1947), 35-79, 404-44; X (1949), 3-40, 359-93.

²⁶*Ibid.*, VII (1946), 386-88; *Insight*, pp. 309-12, 395.

²⁷*Insight*, pp. 490-97; *Divinarum Personarum Conceptio Analogica* (Rome: Gregorian Press, 1957), pp. 272-96.

²⁸*Theological Studies*, VIII, 36-46;

Insight, pp. 304-16, passim; cf. also P. Hoenan, "De Origine Primorum Principiorum Scientiae," *Gregorianum*, XIV (1933), 153-84; XIX (1938), 498-514; XX (1939), 19-54, 321-50.

²⁹*Insight*, pp. 58-62.

³⁰*Ibid.*, chaps. 1 and 2, passim; pp. 146-47, 645-47.

³¹*Ibid.*, pp. 311-13.

³²*Metaphysics I*, 982a5-10.

³³*Critique of Pure Reason*, Introduction, Sec. 3.

ophy of mathematics. Hence I would consider as inadequate various other approaches, ranging from theories that treat philosophy as an abstract deductivism to the view that considers philosophy to be a matter of common-sense discussion. I would exclude also systems which enthrone philosophy over science as omniscience guiding ignorance, or which profess mysterious insight into the nature of number and of the continuum which the mathematician cannot attain. Further—and this is the point of most interest to physicists—I would consider that it is precisely the absence of this methodology in the role of philosophy of physics that is at the root of current confusion regarding the nature of both relativity and quantum theory.

No doubt there will be those who resent my restrictive and exacting delineation of the philosopher's task. But it would seem that the goal of the philosopher, of the lover of wisdom, should be wisdom. Further, it would seem that the history of philosophy is the history of a dialectic movement towards that wisdom. And if I go on to call this basic methodology "critical wisdom" I do so in order to lay emphasis on the claim that, as a fundamental component of human wisdom, this continual explicitation of cognitional structure, forced on us by science and mathematics, supplies a genuine answer to Aristotle's question regarding the wise man who should know yet not know all science,³² to Descartes's quest for a method of rightly conducting reason, and to Kant's search for a science which should determine a priori the possibilities, principles, and extent of human knowledge.³³