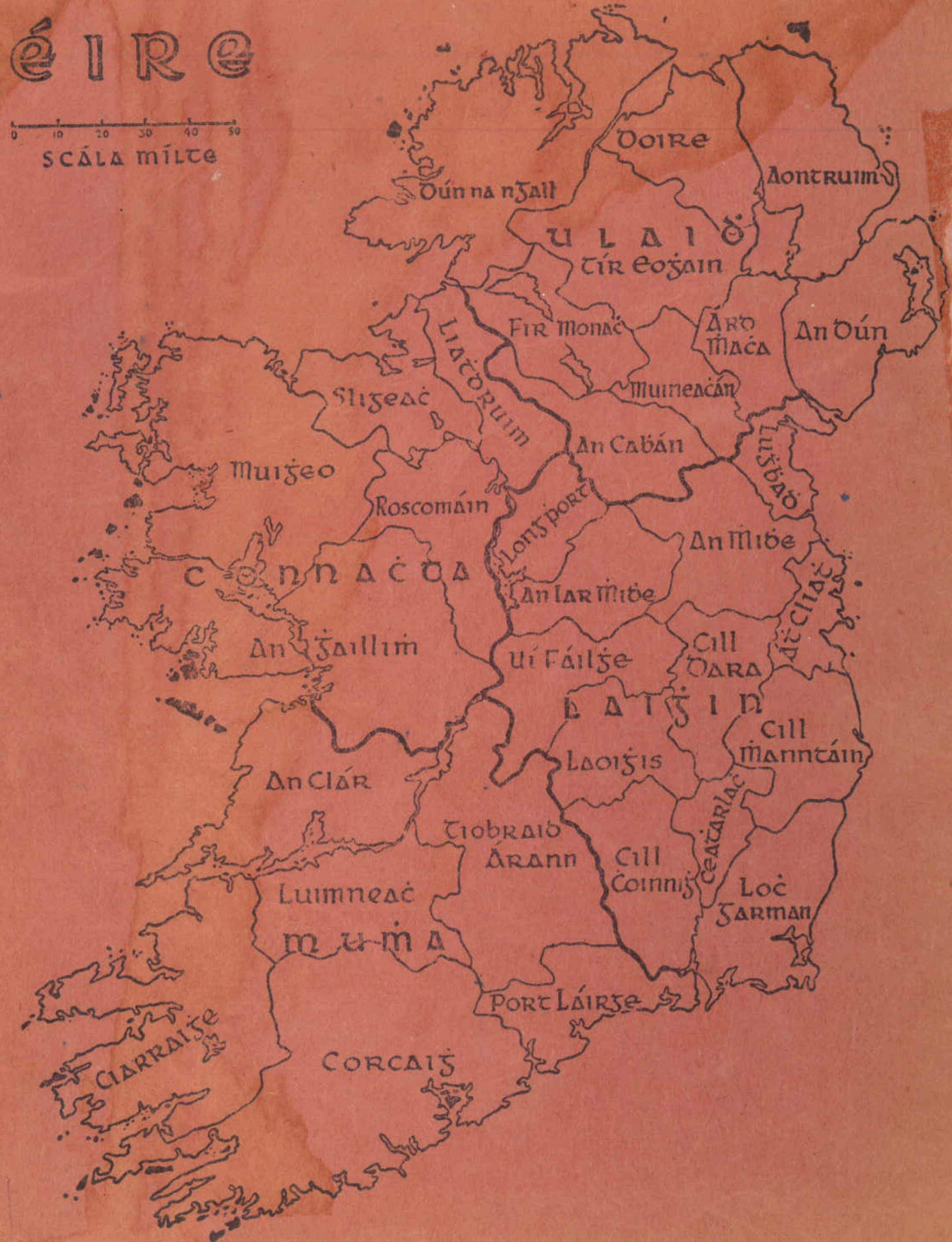


1960

Ih M.P. II

Capital

STANDARD EXERCISE BOOK

CÓIPLEABAR CAIĠDEÁNAĊ

13 : 1953

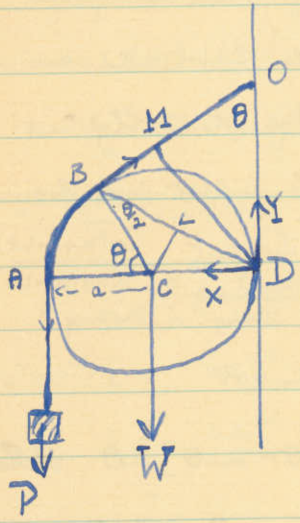
Philip McShane S.J. 35 Lr. Leeson St.

1st yr. Math Phy. Hons. No. 2. (Dynamics).

A12

32 LEAVES

Loney. p. 285 no 6. $\boxed{64}$ "a circle rests in a vertical plane, being pressed against a perfectly ~~smooth~~ rough wall by a string fixed to a pt. in the wall above the circle. The string sustains a weight P, and the coefficient of friction between the string and the circle is μ. If W be the weight of the circle and θ the angle between the string and the wall, show that, if the circle is on the pt. of sliding, then $P(1+\cos\theta)e^{\mu\theta} = W + 2P$.



We have [Cf. Example A. Lect. XXII]

$T_A = P, \quad T_B = Pe^{\mu\theta}$

Moments about D for the forces on the circle give

$$Pe^{\mu\theta}.\,MD = W.a + P.2a.$$

i.e

$$Pe^{\mu\theta}(1+\cos\theta) = W + 2P.$$

$\widehat{ACB} = \widehat{CBL} + \widehat{CDL} = 2\widehat{CBL}$

$\therefore \hat{\theta} = 2\,\widehat{CBL}$

$\therefore \widehat{CBL} = \theta/2. \quad \therefore \widehat{MBL} = \pi/2 - \theta/2$

$MD = BD\,\sin(\pi/2 - \theta/2)$

$= 2BL\,\sin(\pi/2 - \theta/2)$

$= 2a\cos\theta/2\,\sin(\pi/2 - \theta/2)$

$= 2a\,\sin^2(\pi/2 - \theta/2)$

$= a[1 - \cos(\pi - \theta)]$

$= a[1 + \cos\theta]$

evaluate the integral $\int e^{-\mu\theta}(\mu\cos\theta - \sin\theta)\,d\theta$. This integral turns up regularly in these problems so we will calculate its 2 parts in a general way.

Lect. XXIV. Weds Dec 9th

~~In other words we want~~ to evaluate $\overset{I:}{\int e^{ax}\cos bx\,dx}$ and $\overset{J:}{\int e^{ax}\sin bx\,dx}$.

$$\int u\,dv = uv - \int v\,du \qquad \left[\int \frac{d}{dx}[uv]\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \text{ gives formula}\right]$$

I. put $u = \cos bx$, $dv = e^{ax}dx$; then $v = \frac{e^{ax}}{a}$, $du = -b\sin bx\,dx$

$$I = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$

$$J = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}I$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}J.$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b e^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}I$$

So $(a^2+b^2)I = ae^{ax}\cos bx + be^{ax}\sin bx$

$$I = \frac{ae^{ax}\cos bx + be^{ax}\sin bx}{a^2+b^2}$$

$$J = \frac{e^{ax}}{a}\sin bx - \frac{be^{ax}\cos bx}{a^2} - \frac{b^2}{a^2}J$$

$$(a^2+b^2)J = ae^{ax}\sin bx - be^{ax}\cos bx$$

$$J = \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2+b^2}$$

Now we had $T_B e^{-\frac{\mu\pi}{2}} = aw\int_0^{\pi/2} e^{-\mu\theta}(\mu\cos\theta - \sin\theta)\,d\theta$ $\qquad b=1,\ a=-\mu$

$$= aw\left\{\mu\left[\frac{-\mu e^{-\mu\theta}\cos\theta + e^{-\mu\theta}\sin\theta}{1+\mu^2}\right] - \left[\frac{-\mu e^{-\mu\theta}\sin\theta - e^{-\mu\theta}\cos\theta}{1+\mu^2}\right]\right\}_0^{\pi/2}$$

$$= \frac{aw}{1+\mu^2}\left\{\mu e^{-\frac{\mu\pi}{2}} + \mu e^{-\frac{\mu\pi}{2}} + \mu^2 - 1\right\}$$

$$T_B = \frac{aw}{1+\mu^2}\left[2\mu + (\mu^2-1)e^{+\frac{\mu\pi}{2}}\right]$$

and this tension will support a length $\frac{2\mu a + a(\mu^2-1)e^{\frac{\mu\pi}{2}}}{1+\mu^2}$

Loney p. 286 no. 10. Upon a rough circle fixed vertically is placed a string which subtends an angle β at the centre. If the string is on the pt. of slipping off, prove that the angular distance α of its upper end from the highest pt. of the circle is determined by the eqt. $\frac{\cos(\alpha+\beta-2\epsilon)}{\cos(\alpha-2\epsilon)} = e^{\beta\tan\epsilon}$ where ε is the angle of friction, and α is measured in the direction towards which the string is slipping.

~~as in the last problem we got~~

$$Te^{-\mu\theta} = \int_\alpha^{\alpha+\theta} awe^{-\mu\theta}[\mu\cos\theta - \sin\theta]\,d\theta$$

and we get the result by using the conditions: $\{T=0 \quad \theta = \alpha+\beta\}$

$$0 = \frac{aw}{1+\mu^2}\left\{2\mu e^{-\mu\theta}\sin\theta + (1-\mu^2)e^{-\mu\theta}\cos\theta\right\}_\alpha^{\alpha+\beta}$$

$$0 = -\frac{awe^{-\mu\alpha}}{1+\mu^2}\left\{2\mu\sin\alpha + (1-\mu^2)\cos\alpha\right\}$$

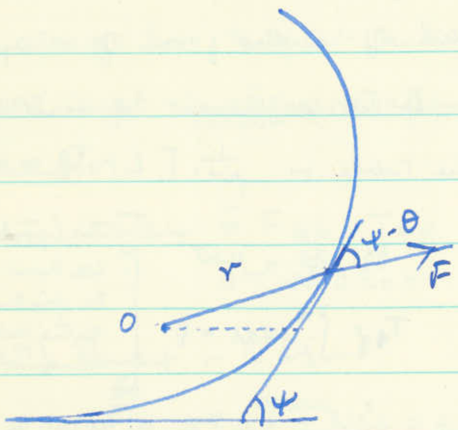
$$+\frac{awe^{-\mu\alpha-\mu\beta}}{1+\mu^2}\left\{2\mu\sin(\alpha+\beta) + (1-\mu^2)\cos(\alpha+\beta)\right\}$$

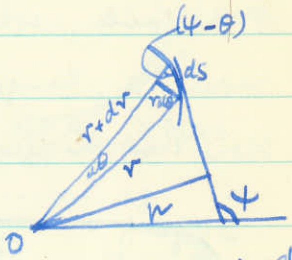
$$2\mu\sin\alpha + (1-\mu^2)\cos\alpha = e^{-\mu\beta}\left[2\mu\sin(\alpha+\beta) + (1-\mu^2)\cos(\alpha+\beta)\right]$$

or $e^{\beta\tan\epsilon} = \frac{2\tan\epsilon\sin(\alpha+\beta) + (1-\tan^2\epsilon)\cos(\alpha+\beta)}{2\tan\epsilon\sin\alpha + (1-\tan^2\epsilon)\cos\alpha}$ (divide above and below by "$1-t^2\epsilon$")

$= \frac{\tan 2\epsilon\sin(\alpha+\beta) + \cos(\alpha+\beta)}{\tan 2\epsilon\sin\alpha + \cos\alpha}$ (multiply above & below by $\cos 2\epsilon$) $= \frac{\cos(\alpha+\beta-2\epsilon)}{\cos(\alpha-2\epsilon)}$

due to the action of a central force (i.e. a force "radiating" from a point: O) F, force per unit mass, = f(r)





$\cos(\psi-\theta)=\frac{dr}{ds}$

$\sin(\psi-\theta)=\frac{p}{r}$

For equil. i) $\frac{dT}{ds} = -F_m\cos(\psi-\theta) = -F_m\frac{dr}{ds}$

ii) $T\frac{d\psi}{ds} = F_m\sin(\psi-\theta)$

or $T = F_m\rho\sin(\psi-\theta)$ $\qquad \frac{ds}{d\psi}=\rho = r\frac{dr}{dp}$

$= F_m \cancel{r}\frac{dr}{dp}\cdot\frac{p}{\cancel{r}}$

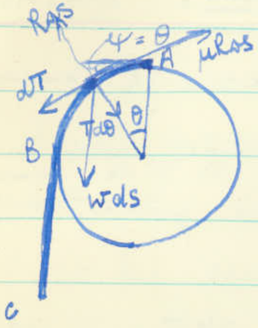
i) gives $T = -\int F_m dr$ + const. A (to be det. from initial condns) (α)

dividing i by ii) $\frac{dT}{T} = -\frac{dp}{p} \rightarrow \log T+\log p$ = const. $\rightarrow T.p = B$ say. (β)

I given the force: we get T from (α) then (p,r) eqt. of curve from (β)

II given the curve: $\rightarrow (p,r)$ eqt. then (β) gives T and $F = -\frac{1}{m}\frac{dT}{dr}$ gives $F(r)$.

C. { Loney. p286. no. 11. "a uniform heavy string rests on the upper surface of a rough vertical circle of radius a, and partly hangs vertically. Prove that, if one end be at the highest point of the circle, the greatest length that can hang freely is $\frac{2\mu a+(\mu^2-1)ae}{\mu^2+1}$.



Let wt. per unit length be w.

[Note: ψ = θ; also we measure s from top.] { about to slip down

we have $dT + w\,ds\sin\theta - \mu R\,ds = 0$

1) $\frac{dT}{ds} = \mu R - w\sin\theta$

also $T\,d\theta + w\,ds\cos\theta - R\,ds = 0$

2) $T\frac{d\theta}{ds} = R - w\cos\theta$

we eliminate R between 1) and 2)

$\frac{dT}{ds} - \mu T\frac{d\theta}{ds} = \mu w\cos\theta - w\sin\theta$

$\times\frac{ds}{d\theta}$ or $\frac{dT}{d\theta} - \mu T = w\frac{ds}{d\theta}(\mu\cos\theta-\sin\theta)$ $\qquad$ but $s = a\theta$ $\therefore \frac{ds}{d\theta}=a$

$= aw(\mu\cos\theta - \sin\theta)$

[we met this differential eq before: $e^{\int P dx} = e^{-\mu\theta}$

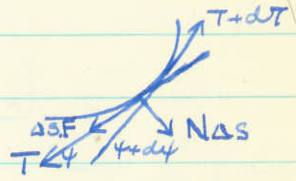
$\frac{d}{d\theta}[Te^{-\mu\theta}] = awe^{-\mu\theta}(\mu\cos\theta-\sin\theta)$

$Te^{-\mu\theta} = \int_0^\theta awe^{-\mu\theta}(\mu\cos\theta-\sin\theta)d\theta + C$ $\qquad$ C=0 since T=0 (at A) when θ=0

$T_Be^{-\frac{\mu\pi}{2}} = aw\int_0^{\pi/2} e^{-\mu\theta}(\mu\cos\theta-\sin\theta)d\theta$

This equation will give us the Tension at B: and this tension will be equal to the wt. of the unknown free length which we can then calculate. So our only difficulty is to

Next we want to consider the 2nd form of the equil. eqns: i.e. we take the ~~ext~~ components of external forces on the string acting along the tangent and normal: F, N per unit length (more useful form, as we shall see).



1) $(T+\Delta T)\cos\Delta\psi - F\Delta s - T = 0$

$\Delta T - F\Delta s = 0 \quad \to \quad \frac{dT}{ds} = F$

2) $(T+\Delta T)\sin\Delta\psi - N\Delta s = 0$

$T\Delta\psi + \Delta T.\Delta\psi - N\Delta s = 0 \quad \to \quad T\frac{d\psi}{ds} = N$ $\qquad \frac{d\psi}{ds} = \frac{1}{\rho}$.

Examples of where this form turns up:

A. "Holding a boat": Then $N = R$; $F = \mu R$ (boat "about to" slip away).

1) $\frac{dT}{ds} = \mu R$

2) $\cancel{\mu}T\frac{d\psi}{ds} = R$

1 − 2) × μ: $\frac{dT}{ds} - \mu T\frac{d\psi}{ds} = 0$

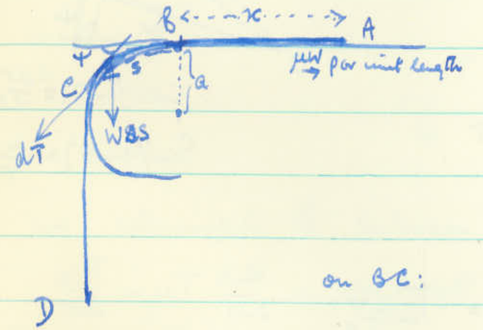
$\frac{dT}{T} = \mu\, d\psi$

$\log T = \mu\psi + \log T_0$ (measure ψ from the pt. where the rope touches the post at your end.)

$T = T_0 e^{\mu\psi}$ (put in numbers.)

Mon. 7th Dec '59. Lect. XXIII

A. Loney. p. 286. no. 12 "A heavy chain of length l; rests partly on a rough table and the remainder after passing over the smooth edge of the table, which is rounded off in the form of a cylinder of radius a, hangs freely down. If the coefficient of friction is μ, show that the least length on the table is $\frac{1}{\mu+1}\left[l - \frac{\pi a}{2} + a\right]$.



AB: $\frac{dT}{ds} = \mu w$

$T_B = \int_0^x \mu w\, ds + c$

$= \mu w x$

[The problem is to see how much support the friction on the surface of the table gives the hanging end of the string]

[let w be wt. p. unit length]

on BC: $\frac{dT}{ds} + w\sin\psi = 0$

$\frac{dT}{d\psi} = -aw\sin\psi$ $\qquad$ since $\frac{ds}{d\psi} = a$.

$T_C = \int_0^{\pi/2}[-aw\sin\psi\, d\psi] + T_B$

$= aw\cos\psi\Big|_0^{\pi/2} + \mu w x$

also $T_C = w(l - \frac{\pi a}{2} - x)$

$l - \frac{\pi a}{2} - x = -a\cancel{w} + \mu x$

$l - \frac{\pi a}{2} + a = (\mu+1)x$.

[omit section B. for the moment: too complicated.]

B. There is one special type of problem when the string is not on a surface, and yet in which a modified form of Form b) of the equilibrium equations gives an easy method: that is, in the case when the string is in equilibrium equations: for, if the chain is tightly stretched the sag is small and c is large (T is large)

so

1) $y_Q = h + c = c \cosh \frac{d}{c}$

$$= c\left[1 + \frac{d^2}{2c^2} + \frac{d^4}{4!c^4} + \cdots\right]$$

$$\doteq c + \frac{d^2}{2c} \qquad \left(\text{if } c \text{ large, } \frac{d^4}{24c^4} \text{ v. small etc.}\right)$$

so $1)^A$ $\underline{h = \frac{d^2}{2c}}$ is our approximate relation instead of 1).

2) $l = c \sinh \frac{d}{c}$

$$= c\left[\frac{d}{c} + \frac{d^3}{3!c^3} + \frac{d^5}{5!c^5} \cdots\right]$$

$$\doteq d + \frac{d^3}{6c^2}$$

so $2)^A$ $\underline{l - d = \frac{d^3}{6c^2}}$ is our approximate relation instead of 2).

Regarding the use of these formulae: Cf. 1958 paper

(i) If d is neither given nor wanted use 3)

(ii) If h is given or wanted use 1) (or $1)^A$)

(iii) If l is given or wanted use 2) (or $2)^A$)

Notice, that when c is large

$$y = c \cosh \frac{x}{c} = c\left[1 + \frac{x^2}{2c^2} + \frac{x^4}{4!c^4} \cdots\right]$$

$$\doteq c + \frac{x^2}{2c}$$

$y - c = \frac{x^2}{2c}$ is the equation of the string: a parabola [change origin to $(0, c)$ to see this clearly: $y' = \frac{x^2}{2c}$]

The case c large is of course the most important case (telegraph wires for example).

Finally some numerical examples: e.g.

"~~Two tensions~~, each of 75 lbs. wt, support a uniform chain, 100 ft. long, weighing 1 lb. per foot. Calculate sag."

(note: d is neither given nor wanted!)

For suspension bridge  mechanics eqts: $Y = -\lambda \frac{dx}{ds}$, $X = 0$.

we get $\frac{d}{dx}\left(T_0 \frac{dy}{dx}\right) = m\lambda$

$T_0 \frac{dy}{dx} = m\lambda x + \cancel{C}$ $\quad x = 0,\ \frac{dy}{dx} = 0$

$T_0 y = \frac{m\lambda x^2}{2} + B$

$y = c\cosh\frac{x}{c}$ we see that the bigger the Tension (and therefore c) the smaller the sag [and vice versa: this relationship is physically obvious], because $\frac{x}{c}$ changes less with x if c is big: and therefore y also changes more slowly.

Now we have $y = c\cosh\frac{x}{c}$ and $s = c\sinh\frac{x}{c}$: and since $1 + \sinh^2\frac{x}{c} = \cosh^2\frac{x}{c}$ we have

$$1 + \frac{s^2}{c^2} = y^2/c^2 \quad \text{or} \quad y^2 = c^2 + s^2.$$

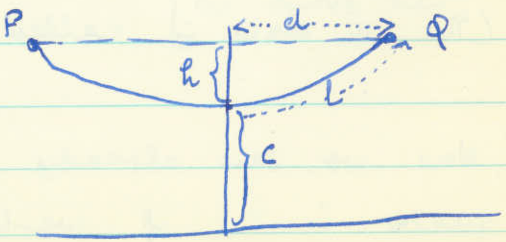
We have here the three basic equations from which we can get all the information we want about the catenary:

1) $y = c\cosh\frac{x}{c}$ 2) $s = c\sinh\frac{x}{c}$ 3) $y^2 = c^2 + s^2$

If we have a chain (rope etc) hanging from 2 points P, Q on the same horizontal level, we call the vertical distance between P (or Q) and the lowest point the sag $= h$, say.

Notice that

$h + c = y$ coord. of P.



[note: the distance $2d$ we generally call the span: and we will use $2l$ for the length of the chain]

1)	2)	3)
$y = c\cosh\frac{x}{c}$	$s = c\sinh\frac{x}{c}$	$y^2 = c^2 + s^2$
$\downarrow$	$\downarrow$	$\downarrow$
$h + c = c\cosh\frac{d}{c}$	$l = c\sinh\frac{d}{c}$	$(h+c)^2 = c^2 + l^2$
$\downarrow$	$\downarrow$	
given any 2 of sag, c, and x-coordinate of end pt. we can get the other accurately.	given any 2 of length, c, and x coord. of end pt (i.e. $\frac{1}{2}$ the span) we can get the other accurately.	given $h, l \xrightarrow{\text{we can get}} c$ (or $y, l \rightarrow c$)
$c, d \rightarrow h$	$c, d \rightarrow l$	$c, l \rightarrow h$
$c, h \rightarrow d$	$c, l \rightarrow d$	$c, h \rightarrow l$
$d, h \rightarrow c$	$d, l \rightarrow c$	accurately and immediately.

Lect. XXII Fri. Dec 4th '59

For the first two equations however we do not get an immediate numerical value: we generally have to find the value of a hyperbolic fc. However in the special circumstance of a tightly stretched rope we can get very good approximate results instead of the first two

we have 2 arbitrary constants: if we choose the coordinate axes so that when $x=0$ $\frac{dy}{dx}=0$ then $B=0$
(because $\sinh 0 = 0$)

so $$\frac{dy}{dx} = \cancel{\tfrac{A}{mg}}\ \sinh\left(\frac{mg}{A}x\right)$$

and therefore $$y = \frac{A}{mg}\cosh\left(\frac{mg}{A}x\right).$$

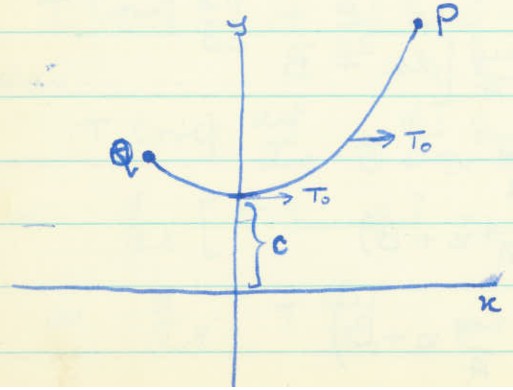
Now what is A?
we have $T\frac{dx}{ds} = A$ i.e. the horizontal component of the tension constant. If we let the tension at the lowest pt. be T_0 (i.e:– ) then $A = T_0$

$$y = \frac{T_0}{mg}\cosh\left(\frac{mg}{T_0}x\right)$$

For convenience we take T_0 equal to cmg (c therefore a constant) i.e. equal to the weight of a length c of the rope, so that

$$y = c\cosh\left(\frac{x}{c}\right)$$

notice that at the lowest pt, $x=0$ and $y=c=\frac{T_0}{mg}$: so the bigger the tension the further up from the x-axis the curve lies.



Now we have the eqt. of the "catenary": we would also like the tension at any pt: we have an eqt. for T: $T\frac{dx}{ds} = T_0$, i.e. $T = T_0\frac{ds}{dx}$ but we haven't got s.

We get s from the eqt. $\frac{d}{ds}\left(T_0\frac{dy}{dx}\right) = mg$ cf. 2 pages back.

This gives $\frac{d}{ds}\left(c\frac{dy}{dx}\right)=1$

i.e. $c\frac{dy}{dx} = s + \cancel{K}$ ← measure s from $\frac{dy}{dx}=0$.

so $$s = c\,.\,c\,.\,\frac{\sinh\frac{x}{c}}{c} = c\sinh\frac{x}{c}.$$

So the tension at any pt. (x', y'), given by

$$T = T_0\frac{ds}{dx}\Big|_{\text{at } x'y'} \text{ is } T_0\, c\,\frac{d}{dx}\left[\sinh\frac{x}{c}\right]_{\text{at } x'y'} = T_0\cosh\frac{x'}{c}$$

$$= c\cosh\frac{x'}{c}\,.\,mg$$

$$= y'\,.\,mg.$$

So the tension at any pt. $(x'y')$ is equal to the weight of a piece of the rope which if hung by one end at the pt. would reach the x axis. (The x axis is called the directrix).

Now we saw already that the bigger the tension the greater is c: if we look again at the equation

$\rightarrow$ $(T+\Delta T)\cos(\varphi+\Delta\varphi) + m\Delta s.X - T\cos\varphi = 0.$

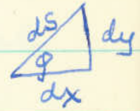
$(T+\Delta T)[\cos\varphi\cos\Delta\varphi - \sin\varphi\sin\Delta\varphi] + m\Delta s.X - T\cos\varphi = 0$ $\left\{\begin{array}{l}\cos\Delta\varphi \simeq 1\\ \sin\Delta\varphi \simeq \Delta\varphi\end{array}\right.$

$\cancel{T\cos\varphi} - T\sin\varphi.\Delta\varphi + \Delta T\cos\varphi - \Delta T\sin\varphi.\Delta\varphi + m\Delta s.X - \cancel{T\cos\varphi} = 0$

$-T\sin\varphi\frac{\Delta\varphi}{\Delta s} + \frac{\Delta T}{\Delta s}\cos\varphi - \frac{\Delta T}{\Delta s}\sin\varphi.\Delta\varphi + mX = 0.$

letting $\Delta\varphi \to 0$ ($\therefore \Delta T$ and $\Delta s \to 0$) we get

$$-T\sin\varphi\frac{d\varphi}{ds} + \frac{dT}{ds}\cos\varphi + mX = 0$$



or $\frac{d}{ds}(T\cos\varphi) + mX = 0$

or $\frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0.$ - - - - (1)

likewise for $\uparrow$ resolution:

$(T+\Delta T)\sin(\varphi+\Delta\varphi) + m\Delta s.Y - T\sin\varphi = 0$

$(T+\Delta T)[\sin\varphi.1 + \cos\varphi.\Delta\varphi] + m\Delta s.Y - T\sin\varphi = 0$

$\cancel{T\sin\varphi} + T\cos\varphi.\Delta\varphi + \Delta T.\sin\varphi + \Delta T\cos\varphi.\Delta\varphi + m\Delta s.Y - \cancel{T\sin\varphi} = 0$

$T\cos\varphi\frac{\Delta\varphi}{\Delta s} + \frac{\Delta T}{\Delta s}\sin\varphi + \frac{\Delta T}{\Delta s}\cos\varphi.\Delta\varphi + mY = 0$

$\Delta\varphi \to 0$

$$T\cos\varphi\frac{d\varphi}{ds} + \frac{dT}{ds}\sin\varphi + mY = 0$$

or $\frac{d}{ds}[T\sin\varphi] + mY = 0$

or $\frac{d}{ds}\left[T\frac{dy}{ds}\right] + mY = 0$ - - - - (2)

[Note: it is worth while trying to get the "physical significance of these eqts: They merely state the rate at which the tension changes as you go up the string is such that the "bit" of external force is balanced: hence equilibrium]

Lect. XXI Weds. Dec 2nd

Example: $Y = -g \quad X = 0.$ a Rope suspended from 2 pts.

our 2 eqts. become

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0 \quad \text{and} \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right) - gm = 0$$

$\therefore T\frac{dx}{ds} = A$ (a const.) [physical significance: The X component of tension constant: which is obvious. why?]

put this value of T in second eqt:

$$\frac{d}{ds}\left(A\frac{ds}{dx}.\frac{dy}{ds}\right) = mg$$

or $\frac{d}{dx}\left(A\frac{dy}{dx}\right) = mg\frac{ds}{dx}$

$$A\frac{d^2y}{dx^2} = mg\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

to integrate this eqt. put $v = \frac{dy}{dx}$

$$A\frac{dv}{dx} = mg\sqrt{1+v^2}$$

$$\therefore \frac{dv}{\sqrt{1+v^2}} = \frac{mg\,dx}{A}$$

integrating we get $\int\frac{dv}{\sqrt{1+v^2}} = \frac{mg}{A}\int dx + B$

substitute $v = \sinh(u) \rightarrow$

$$\sinh^{-1}v = \frac{mg}{A}x + B$$

or $v = \sinh\left(\frac{mg}{A}x + B\right)$

i.e. $\frac{dy}{dx} = \sinh\left(\frac{mg}{A}x + B\right)$

Lect. $\overline{XX}$. Mon Nov 30th. Equilibrium of Strings (flexible)

What new idea involved here? That we have a structure which is capable of supporting only tension forces: no bending moment (it does not resist effort to bend it), no shear force (this is not as easy to see but ~~if you~~ consider that the string is able to take up such a shape that its reaction to external forces at any pt. is along itself.).

2 Types of problem occur here:

(i) The string is free to some extent, and under some external forces: we look for

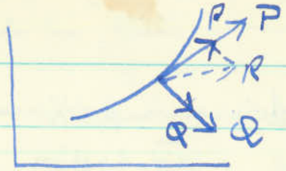
a) its equation b) the Tension at any pt.

(ii) The string is on a surface (or curve): we look for:

a) the condition(s) of equilibrium [Reactions, etc] b) the tension at any pt.

In order to be able to tackle these 2 types of problem more easily we work out the general equations of equilibrium for a string in two ways: one form being suited to the first type of problem, the other to the second.

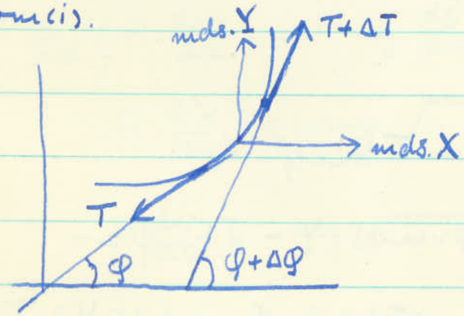
Form (i)



Form (ii)

We get the 2 different forms merely by taking different components of the external resultant force R. They are really then not different equations but (as will be seen shortly) Form (i) is more suitable for Problem (i), Form (ii) for problem (ii).

Form (i).



Let X, Y be the forces per unit mass: ∴ if m is the mass per unit length the force on Δs is $m\Delta s . X \rightarrow$, $m\Delta s . Y \uparrow$. Taking Δs very small we can say that this piece Δs is in equil. under the tensions T and $T+\Delta T$ and forces $m\Delta s . X$, $m\Delta s . Y$.

So resolving → and ↑ we get the usual 2 eqts. [notice, no moment eqt.]

Lect. XIX

Do D first then A.

A. The Intrinsic Eqt. of a curve is the eqt. of the curve expressed in terms of the length of the curve s at P from a given point Q, and of the angle ~~beta~~ which the tangent at P makes with the tangent at Q

e.g. circle:  $s = a\psi$.

[B. The Polar equation of a curve (r, θ): Eqt. in ~~terms~~ only mention of ~~dist.~~ length of "radius vector" from fixed pt. and of the angle between " " & fixed line.]

$\frac{l}{r} = 1 + e\cos\theta$ 

C. $\int e^{ax}\cos bx\,dx \qquad \int e^{ax}\sin bx\,dx$

D. Differential Eqts:

(i) $\frac{d^2y}{dx^2} = f(y)$ we cannot integrate this directly $\int \frac{d^2y}{dx^2}dx = \int f(y)\,dx$ no good.

but we notice that $2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]$

and trying this (we multiply across by it & integrate $\int dx$)

we find $\int \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]dx = \int 2f(y)\frac{dy}{dx}\,dx + C$

$= \int 2f(y)\,dy + C.$

$\left(\frac{dy}{dx}\right)^2 = F(y) + C$

if $\frac{dF}{dy} = 2f(y)$.

$\frac{dy}{dx} = {}^{+r}\sqrt{F(y)+C}$

we separate var. & integ: $\int dx = \int \frac{dy}{\sqrt{F(y)+C}}$

$x =$

(ii) $\frac{dy}{dx} + P(x)y = Q(x)$.

(if L.H.S. were $\frac{d}{dx}$[something] then we could integrate: ~~so we multiply across by an unknown fc.~~

~~$g(x)$ i.e. $g(x)\frac{dy}{dx} + P(x)g(x)y$ and try &~~

~~select $g(x)$ so that we have $\frac{d}{dx}[g(x)y]$.~~

Now $\frac{d}{dx}[g(x)y] = g(x)\frac{dy}{dx} + \frac{dg}{dx}(x)\,y$

So if we can find $g(x)$ so that

$\frac{dg}{dx} = P(x)g(x) \qquad g = e^{\int P\,dx}$

So $\times$ across by $e^{\int P dx}$ $\Rightarrow$ sol. by integration conditions: e.g. iii) $EI\frac{d^4y}{dx^4} = w$ has as solution

$$EIy = \frac{wx^4}{24} + Ax^3 + Bx^2 + Cx + D.$$

Now if we take the problem we had, i.e.

we have the following conditions:

1) $y = 0$ $x = 0$ ✓ D

$y = 0$ $x = l$

$\frac{dy}{dx} = 0$ $\begin{cases} x = 0 \ ✓ \ C \\ x = l/2 \\ x = l \end{cases}$

$$EIy = \frac{wx^4}{24} + Ax^3 + Bx^2 + \cancel{C}x + \cancel{D} \quad \underset{x=0}{y'=0} \quad \underset{x=0}{y=0}$$

$$EI\frac{dy}{dx} = \frac{wx^3}{6} + 3Ax^2 + 2Bx$$

$y = 0$, $x = l$:

$$\frac{wl^3}{24} + Al^2 + Bl = 0$$

$y' = 0$, $x = \frac{l}{2}$:

$$\frac{wl^3}{6.8} + \frac{3Al^2}{4} + Bl = 0$$

$$\frac{wl^3}{48} - \frac{1}{4}Al^2 = 0 \qquad A = -\frac{wl}{12}$$

$$-\frac{2}{48}wl^3 = -Bl \qquad B = \frac{1}{24}wl^2$$

~~$E\frac{d^2y}{dx^2} =$~~

so $EI\frac{dy}{dx} = \frac{wx^3}{6} - \frac{wlx^2}{4} + \frac{wl^2x}{12}$

$$EI\frac{d^2y}{dx^2} = \frac{wx^2}{2} - \frac{wlx}{2} + \frac{wl^2}{12}$$

$= 0$ when $6x^2 - 6xl + l^2 = 0.$

$$x = \frac{3 \pm \sqrt{9-6}}{6}\, l$$

$$= \frac{l}{2} \pm \frac{\sqrt{3}}{6}\, l$$

$$EIy = \frac{wx^4}{24} - \frac{wl}{12}x^3 + \frac{1}{24}wl^2x$$

$EIy_{\text{when } x = l/2}$ ✓.

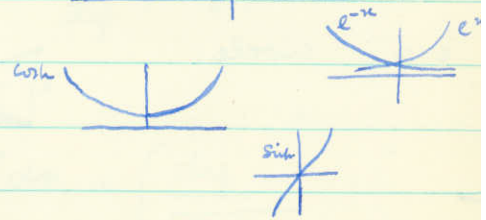
C. (i) Hyperbolic fcs. (ii) $\int \frac{dv}{\sqrt{1+v^2}}$

Define $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\frac{d}{dx}(e^x) = e^x$. $\int e^x dx = e^x$

Define $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$



$\frac{d}{dx}[\cosh x] = \dots \sinh x.$

$\frac{d}{dx}[\sinh x] = \dots \cosh x.$

General Rule: in any trig relation change the sign of any square of "sin" or its equivalent and you will have the hyperbolic relation.

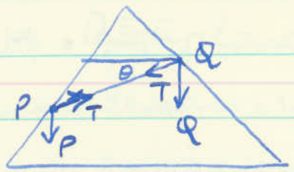
e.g. $\cosh^2 = 1 + \sinh^2$

(ii) $\int \frac{dv}{\sqrt{1+v^2}}$ $\quad v = \sinh x$

$$\int \frac{\cosh x\, dx}{\cosh x} = \int dx = x = \sinh^{-1} v.$$

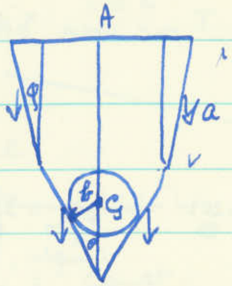
B. 1959. no. 3.

2 wts: string l, λ.



$(P+Q)\tan\theta = P\cot B - Q\cot A$. + find length PQ.

C. 1959. no. 7 V. WK.



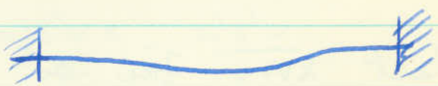
Prove $Wb = a\sin^2\theta\,[(w+W)\tan\theta + (3w+W)\tan\phi]$

where $2\sin\theta = 1 + 2\tan\phi$.

Lect. XVII Nov 23^{rd}.

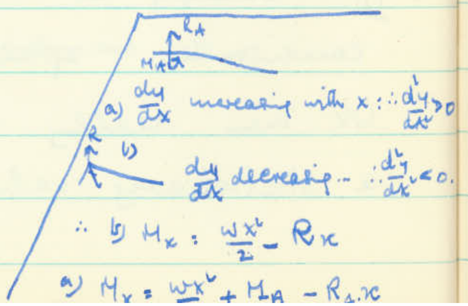
A. Clarify last day's wk.

B. 1949: $\frac{d^4y}{dx^4} = w$. Then

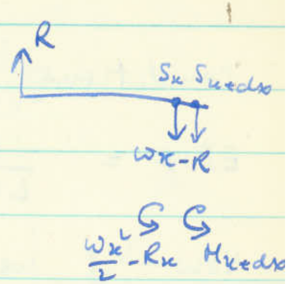


(a) give ~~"new proof"~~ of $\frac{dM}{dx} = S$ $\frac{dS}{dx} = w$.

(b) ~~start problem~~.



Lect. XVIII Nov 25^{th}.



A. (a) Clarify $\frac{dM_x}{dx} = S_x$ $\frac{dS_x}{dx} = w$

B. Begin differential eqts:

Differential eqts. are equations of the type

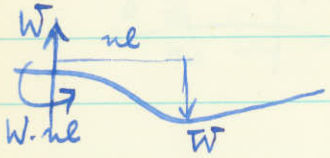
i) $\frac{dy}{dx} = 2x$ ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = \cos x$

iii) $\frac{d^4y}{dx^4} = w$. iv) $\frac{dy}{dx} = -\frac{x}{y}$

To solve a diff. eqt. we have to get y in terms of x i.e. $y = f(x)$ such that when we put this value of y, and the corresponding values of the derivatives, back into the eqt. we get an identity in x: e.g. for i) $y = x^2 + 6$ is a solution because then $\frac{dy}{dx} = 2x$ and putting this value into i) gives us $2x = 2x$. ~~B~~ Note that "6" does not matter $y = x^2 + a$ general sol. In general a solution of a differential eqt. of order n has n arbitrary constant, which have to be determined by particular

Lect XV. Weds 18th Nov.

A. Example: A light lever of length l carries a concentrated load W at a distance nl $(n<1)$, from an end which is clamped horizontally. The other end is free. Prove that the deflection at a distance x from the fixed end is $\frac{Wx^2}{6EI}(3nl-x)$ $x<nl$ or $\frac{Wn^2l^2}{6EI}(3x-nl)$ $x>nl$.



$x<nl$

$EIy'' = M_x = nlW - Wx$

$EIy' = nlWx - \frac{Wx^2}{2} + A$ $\left\{ A=0 \;\because y'=0,\ x=0 \right\}$

$EIy = \frac{nlWx^2}{2} - \frac{Wx^3}{6}$

$x>nl$

$EIy'' = nlW - Wx + W(x-nl) = 0$

$EIy' =$ const.

$\left(EIy' = \frac{n^2l^2W}{2} \text{ when } x=nl, \text{ so}\right)$

$EIy' = \frac{n^2l^2W}{2}$

$EIy = \frac{n^2l^2Wx}{2} + K$

$\left(x=nl,\ EIy = \frac{n^3l^3}{3}W \text{ so } K = -\frac{n^3l^3W}{6}\right)$

$EIy = \frac{n^2l^2Wx}{2} - \frac{n^3l^3W}{6}$

B. Consideration of difficulties $\frac{d^3y}{dx^3}$ discontinuous, etc.

C. Outline of proof of $\frac{EI}{\rho} = M$. (slightly elas. horiz.; horiz. load

I Assumptions (i) Total Tension at any section zero.

(ii) plane sections remain plane sections.

II Proof:

(α) first get T p. unit area = $\frac{Ex}{\rho}$ (using (ii)) at x above neutral plane

(β) ∴ T per area δA_x = $\frac{Ex\,\delta A_x}{\rho}$

(γ) "Total tension zero (i)" $\xrightarrow{\text{only a couple}}$

∴ $\Sigma x\,\delta A_x = 0$

gives the fact that c. of g. of the area lies on G_1G_2 → because we know the pos. of G_1G_2 we can calculate $\Sigma x^2\delta A_x = I$. ∴ $M = \frac{EI}{\rho}$

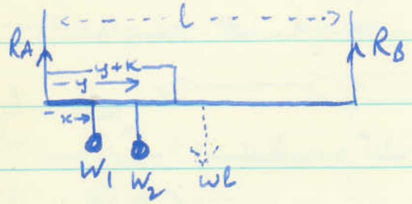
(δ) each T_x above has an equal but opposite T_x below: adding the pairs gives the couple: with moment → $\frac{E}{\rho}\Sigma x^2\delta A_x$.

Lect XVI Fri 20th Nov.

A. Clarify the notion of Wk. fn. & Pot. E. by considering sphere on "hills" + graph ... P. of V. Wk. then really means that the Wk. fn. has a stationary value in equilibrium position.

D. Concluding example:

[~~First: General principle~~: The shear at any pt is the force required to keep either end of the beam in equilibrium if the other ~~were taken~~ away; similarly the couple]



rigid rod, uniform, wt. w. p.u. length.

~~R_A and R_B got from~~ ordinary ~~conditions of equil.~~

$$\left.\begin{array}{l} \text{i)}\ R_A + R_B = W_1 + W_2 + wl \\ \text{ii) moments } A: \\ \quad R_B . l - W_1 x - W_2 y - \frac{wl^2}{2} = 0 \end{array}\right\} \text{giving } R_A \text{ and } R_B.$$

~~at pt.~~ $y+k$:

The shear: $\uparrow\downarrow\ R_A - W_1 - W_2 - w(y+k)$

The couple: $R_A(y+k) - W_1(y+k-x) - W_2 k - w(y+k)\frac{(y+k)}{2}$

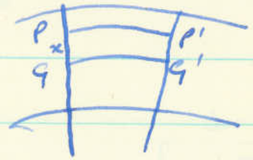
E. preview ~~of next day~~? Cp. XVI homey.

~~Lect XIV.~~ Nov. 16th

A. λ, the modulus of elasticity for the bar: $= E\omega$ where ω the cross section of the bar and E a constant ("Young's modulus") for the material.

Beam bent: ~~without tension~~ slightly. top half extended: bottom contracted:

Neutral "plane" GG' a line in it:



~~Tension per unit area at~~ P:

$$E\,\frac{PP' - GG'}{GG'}$$

$$\rightarrow E\,\frac{x}{\rho}$$

Tension for area $\delta A_x = E\frac{x}{\rho}\delta A_x$

Total $= \sum E\frac{x}{\rho}dA_x$; $= 0$. centroid ~~on neutral line~~.

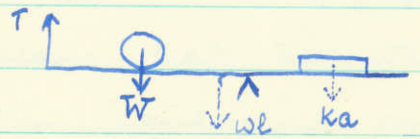
~~moment of couple~~ $\sum E\frac{x}{\rho}dA_x . x \rightarrow \frac{E}{\rho}.\sum x^2 dA \rightarrow \frac{EI}{\rho}$

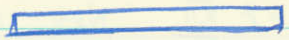
(I ~~moment of~~ inertia ~~of cross~~ section of beam about line thro' centroid $\perp$ length beam.

Slightly flexible i.e. $EI \gg \rho$, $\rightarrow M = EI\frac{d^2y}{dx^2}$.

B. Example of (i) ~~how the above may be put~~ into practice.

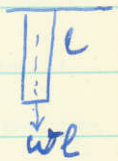
(ii) of rigid beam



Lect XIII Nov 13th 1959.  "Beams"

A. What makes this new group difficult is the introduction of "internal forces" i.e. that if the beam is to "cohere" it must exert internal forces to counteract the external ones: e.g. in the simple case of a pair of forces applied thus:

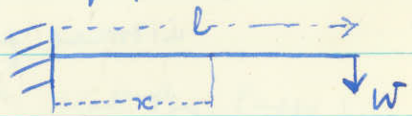
the beam exerts a tension along its length.

A good "gravity" example is the following: 

where the tension increases as we go up the beam.

B. 2 particular examples:

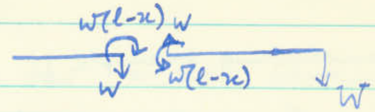
(i) a ~~clamped~~ rigid rod, weight negligible, with a weight hanging from the end:



[Note: a) explain "Rigid" b) explain effect of clamp i.e. that it introduces an extra couple as well as a force, , to be determined]

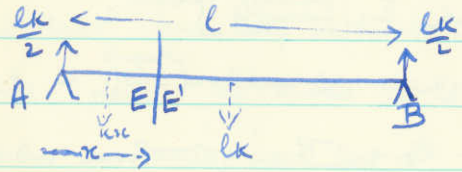
Reaction at wall: ↑W and ↺W.l

At a pt. x: The action of the "wall end" on the rest must be? W↑ and couple W(l−x).]

So at x the situation is: 

So the shear is const. all along, = W

and the "Bending Moment" is W(l−x).

(ii) 

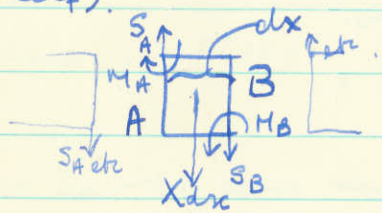
Rigid rod, uniform, wt. k per unit length

The action of BE' must keep AE "in" position": ↑ $kx - \frac{lx}{2}$

↺ $\frac{lk}{2}.x - kx.\frac{x}{2}$

C. Note: in last example: if differentiate M we get S: we expect some relation (from experience of loading, bending, etc.).

We go on to establish general relations between the Bending moment, the Shearing Force, and the "load" (usually the weight itself).



X load p.u.length (uniformly distributed)

For Equilibrium of small section.

1) $S_A - S_B - X dx = 0$

let $S_B = S_A + dS$: $\frac{dS}{dx} = -X$ [signs etc only convention

moments about B:

2) $M_A - M_B + S_A.dx - Xdx.\frac{dx}{2} = 0$

let $M_B = M_A + dM$: $-dM + S.dx - X dx.\frac{dx}{2} = 0$

$dx \to 0$

$-\frac{dM}{dx} = -S$.

than one independent displacement (cf. e.g. Ramsey §11.36(i)) then the equation will lead to more than one equation (by putting all but one displacement equal to zero, then another, etc)

2. When we wish to determine an unknown reaction (tension, thrust, etc...) we choose such a displacement that this reaction does work. Then the reaction will occur in the equation of virtual work, and we get its value in terms of the known forces.

Examples of Type 1: 1958 no 2. Math Phy. Paper 1.
" " " 2: 1950 no 3. " " " 1.
[give in class and begin wh.]

~~[illegible]~~

Sometimes in problem where we have 2 variables there is a geometrical connection; N.B!

Lect XII Nov. 11th '59.

A. Method: a) mark in suitable coordinates, and get the coordinates of the pts. of application of all the forces.

b) get any geometrical connection between the coordinates and lengths etc.

c) in type two consider a tense string, or a thrusting rod as a pair of forces acting in a suitable direction.

B.

~~1956~~. Do 1958 no 2; 1950 no. 3.

C. mention next section:

internal forces on a body [restrict to (i) "rod" and, mostly, (ii) "gravity".]

D. (ii) wk. done by elastic st. in returning to normal:

$$\frac{(m-n)}{2}\left[\frac{\lambda(m-a)}{a}+\frac{\lambda(n-a)}{a}\right]$$

(i) wk. done by gravity in bringing a body down wh.

Define P.E: Wk. that the system would do in bringing it to standard position. (i) P.E wh (ii) elastic string P.E. $\frac{\lambda x^2}{2a}$.

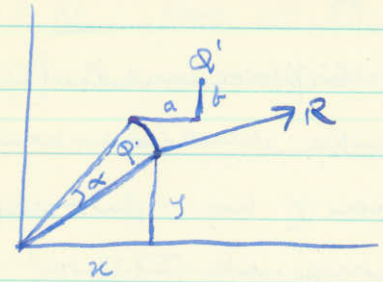
Lect XI. Nov. 9th 59.

The Principle of V. Wk. If for a system of bodies in equilibrium under the action of given external forces and subject to certain constraints it is possible to make a displacement such that the constraining forces do no work, then, for such a displacement, the algebraic sum of the work done by the external forces alone is zero, or of a higher order of smallness than the first in terms of the displacements.

The converse theorem also holds: If a system of bodies is subject to the action of given external forces which are such that, for all small displacements in which the forces of constraint do no work, the algebraic sum of the work done by the external forces is zero, then the system is in equilibrium.

(Note that if any constraining forces are allowed to do work, this work must be included with the work of the external forces, and the theorem still holds.)

Proof:



$Q(xy) \equiv Q(r\cos\theta, r\sin\theta)$

$Q'(x'y') = Q'(r\cos(\theta+\alpha)+a, r\sin(\theta+\alpha)+b)$

$\doteq Q'(x-\alpha y+a, y+\alpha x+b)$

X, Y components of R.

Wk. done: $X(a-\alpha y) + Y(b+\alpha x)$

$= aX + bY + \alpha(Yx - Xy)$

Similarly for all forces $R_2, R_3 \ldots$ Total wk done

$a\sum X_i + b\sum Y_i + \alpha\sum(Y_i x_i - X_i y_i)$.

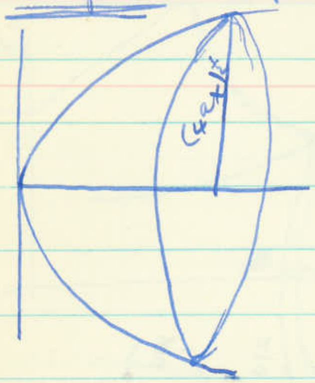
If equil. $\to 0$

If $\to 0$ for all disp. each of a, b, α in turn non-zero $\to$ equil.

There are 2 types of problem where we use this principle:

1) when we wish to determine the position of equilibrium of a system (generally it is a system with only the force of gravity acting on it): in this case we make such a displacement that the constraining forces (reactions etc) do no work: hence we get an equation between the external forces and the coordinates of the system. If there is more

Vol. of Rev: Example: parabola $y^2 = 4ax$.



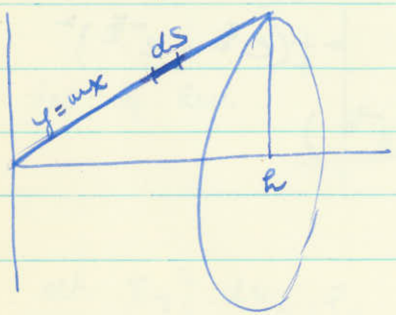
elt. of vol. with C. of G. x_i:

$\pi . 4ax_i . dx_i$

so $\bar{x} = \dfrac{\int_0^h \rho . \pi 4a x_i . x_i dx_i}{\int_0^h \rho . 4a\pi x}$

$= \dfrac{h^3/3}{h^2/2} = \dfrac{2h}{3}$.

Surface of rev.



~~$ds = \sqrt{}$~~

elt. $2\pi y_i . ds_i$

so $\bar{x} = \dfrac{\int_0^h \rho 2\pi y \, ds . x}{\int 2\pi y \, ds}$

$y = mx$

$ds = \sqrt{1+m^2}$

so $\bar{x} = \dfrac{\int_0^h mx^2\sqrt{1+m^2}\,dx}{\int_0^h mx\sqrt{1+m^2}\,dx} \quad (= M)$.

$= \dfrac{h^3/3}{h^2/2} = \dfrac{2h}{3}$.

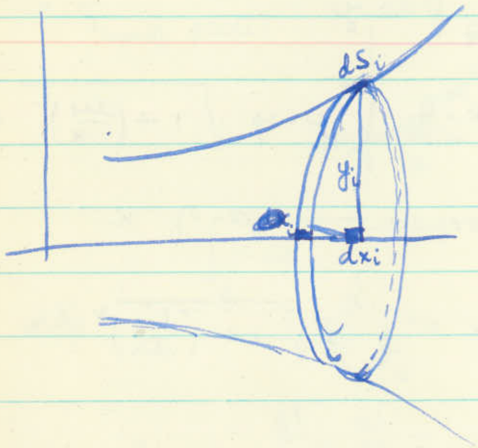
Lect. $\overline{X}$ V. W.

Introductory:

1. The notion of work: a measure of it: F.d.
2. The wk. done by a force = The sum of the works done by its components.
3. 'Plausible' presentation of the fact that the work done by forces in equilibrium in a small displacement from equilibrium is zero.
4. Example of lect. $\overline{VII}$: $F = 2W$, done by this method.

(iii) <u>Surface of Rev.</u>

ρ const.

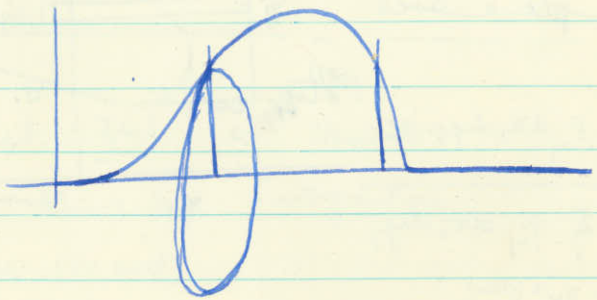


elt:- $dS_i \times 2\pi y_i \times \rho_i$

so $\bar{x} = \dfrac{\int dS . 2\pi y . x}{\int dS . 2\pi y}$

~~$\bar{y} = \int dx . 2\pi$~~ $\bar{y}$ obvious.

(iv) Vol. of Rev.



elt. $\pi y_i^2 . dx_i . \rho_i$.

so $\bar{x} = \dfrac{\int dx . \pi y^2 . x}{\int dx . \pi y^2}$

$e^x = \sum_0^\infty \frac{x^n}{n!}$

$e^{f(x)} \to \frac{d}{dx}(e^{f(x)}) = f'(x) e^{f(x)}$

$\therefore \int e^{f(x)} = \frac{1}{f'(x)} e^{f(x)}$

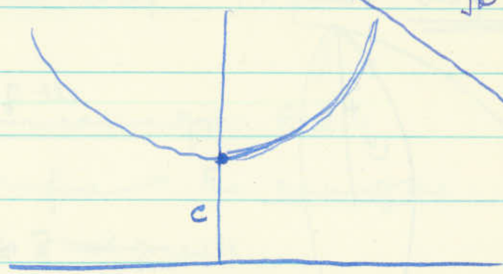
Examples:

(i)^a find $\bar{y}$ for

$y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$

between $x = \pm a$.

ρ const.



$\int_{-a}^{+a} = 2\int_0^a$

$\bar{y} = \dfrac{\int ds . y}{\int ds}$

$\frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}(e^{\frac{2x}{c}} + e^{-\frac{2x}{c}} - 2)$

$= \frac{1}{4}(e^{\frac{2x}{c}} + e^{-\frac{2x}{c}} + 2)$

$= \frac{1}{4}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})^2$

$\int_0^a ds$

$= \int_0^a \frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2} dx$

$= \frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})\Big|_0^a = \frac{c}{2}(e^{a/c} - e^{-a/c})$

$\int_0^a ds . y = \frac{c}{4}\int_0^a \left(\frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{\cancel{2}}\right)^2 dx$

$= \frac{c}{4}\int_0^a (e^{\frac{2x}{c}} + e^{-\frac{2x}{c}} + 2)\, dx$

$= \frac{c}{4}\left[\frac{c}{2}e^{\frac{2x}{c}} - \frac{c}{2}e^{-\frac{2x}{c}} + 2x\right]_0^a$

$= \frac{c}{4}\left[\frac{c}{2}e^{\frac{2a}{c}} - \frac{c}{2}e^{-\frac{2a}{c}} + 2a\right]$

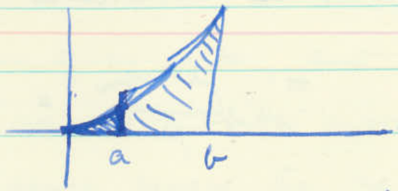
so $\bar{y} = \dfrac{\frac{1}{2}\left[2a + \frac{c}{2}e^{\frac{2a}{c}} - \frac{c}{2}e^{-\frac{2a}{c}}\right]}{e^{a/c} - e^{-a/c}}$

(cf Lamey p14 Ex. 2)

$= \dfrac{a}{e^{a/c} - e^{-a/c}} + \frac{c}{4}(e^{a/c} + e^{-a/c}) = \frac{ca}{2s} + \frac{y_a}{2}$

the fc. at the initial pt. from the value at the end pt.

e.g. area of plate:



$\int_a^b x^2 dx$ rate of ch. at a: $x_a^2 = a^2$

at b: b^2

$\frac{x^3}{3}\Big|_a^b$ Area at a $a^3/3 + C$

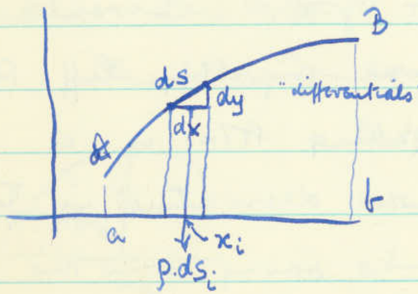
b $b^3/3 + C$.

$\frac{b^3}{3} - \frac{a^3}{3}$

Maths Phy. lect. IX 4th Nov. '59.

C. of G. in general by Integration.

1) of a thin rigid wire (or rod).



$\bar{x} \times \sum \rho ds_i = \sum \rho ds_i . x_i$

in the limit

$\bar{x} \times \int_a^b \rho ds = \int_a^b \rho ds . x$

if ρ const.

then $\bar{x} \times \int_a^b ds = \int_a^b x . ds$

where $ds = \cancel{\text{const}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ and $y = f(x)$ is the eqn. of the curve.

similarly $\bar{y} \times \int_A^B \rho ds = \int_A^B \rho ds . y$

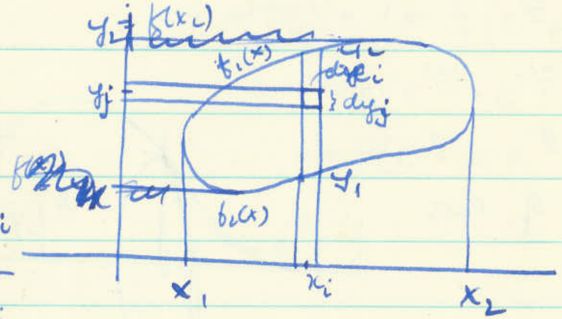
$\bar{y} \times \int_a^b \rho \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \rho . y . \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

where ρ and y are expressed as fcs. of x.

ρ const. $\bar{y} \times \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

[Do "Graph" next]

ii) of a plane area



$$\bar{x} = \frac{\sum_i \sum_j \rho_{ij} dx_i dy_j . x_i}{\sum_i \sum_j \rho_{ij} dx_i dy_j}$$

$$= \frac{\int_{x_1}^{x_2} \int_{y_1 = f(x)}^{y_2 = f(x_2)} \rho dx dy . x}{\int_{x_1}^{x_2} \int_{y_1}^{y_2} \rho dx dy}$$

$$\bar{y} = \frac{\int_{x_1}^{x_2} \int_{y_1}^{y_2} \rho dx dy . y}{\iint \rho dx dy}$$

if ρ const. then take as element $dx_i . (y_2 - y_1)_i$

$$\bar{x} = \frac{\int_{x_1}^{x_2} dx \, (y_2 - y_1) x}{\int_{x_1}^{x_2} dx \, (y_2 - y_1)}.$$

$$\bar{y} = \frac{\int (y_2 - y_1)\left(\frac{y_2 + y_1}{2}\right) dx}{\int (y_2 - y_1) dx}.$$

Now consider $\sum_{i=1\ldots N} y_i \cdot \frac{b}{n}$. it is the area of the sum of the strips: This area is an approx. to the area of the plate, and if we make the strips narrower we get a better approximation:

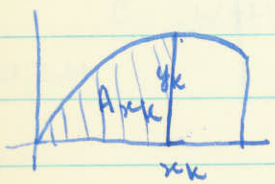
as n becomes larger the no. of strips becomes larger and the width of each becomes smaller ($\sim \frac{b}{n}$). call b/n dx.

we assume (cf. maths course for rigid proof) that this sum has a limit as we let n become large and we call

$$\lim_{n\to\infty} \sum_{i=1\ldots N} \cancel{\frac{y_i b}{n}} = \sum y_i dx_i \text{ where } dx_i = b/n. \quad \text{the "integral"} \int_0^b y\,dx$$

and that if we had the value of this integral it would be equal to the area required. But note that though we have defined this integral we have as yet no means of getting its value.

The following consideration gives us a method.



Let A_{x_K} be the area indicated: we want to examine the rate at which the area A_{x_K} increases as we move forward from x_K. if we move forward a distance ~~to~~ to $x_K + \mu$ then the area increases by $\mu \cdot y_K$. To get the

rate of change of Area at x_K we divide the increase of area by the increase in x.

$$\frac{A(x_K+\mu) - A_{x_K}}{\mu} = \frac{\mu \cdot y_K}{\mu} = y_K.$$

and we see that the rate at which the area increases is measured by the ~~distinct~~ "width of the front advancing" i.e. y_K.

So if we call the rate of change of Area A' then A' at any pt $= y$.

A here ($= \sqrt{2ax - x^2}$

This gives us a method of getting the value of the limit expressed by the integral, because we see that if we had the area at x, $A(x)$ say i.e. as a fc. of x, we would get the fc. under the integral sign by diff. A_x with respect to x. So getting $A(x)$ is a matter of getting the fc. whose derivative is the fc. under the integral sign. Having found this fc., its value ~~at the a pt~~ can only differ by a constant from the required area, and this constant is eliminated by subtracting the value of

We can get this pt. for any system of particles by using a coordinate system, finding the coordinates of each, and using the formulae

$$\bar{x} = \frac{\Sigma w_i x_i}{\Sigma w_i} \qquad \bar{y} = \frac{\Sigma w_i y_i}{\Sigma w_i} \qquad \left(\bar{z} = \frac{\Sigma w_i z_i}{\Sigma w_i}\right)$$

$(w_i = m_i g = \rho_i v g$ etc$)$.

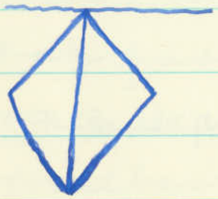
— — — — — —

For some bodies C. of G. can be got immed. because of symmetry.

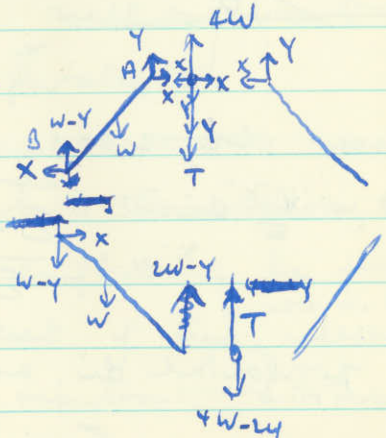
1) rod 2) triangle 3) parallelogram.

— — — — —

B. ~~Discuss Loney~~ p.49 no. 14.



Breakup etc.

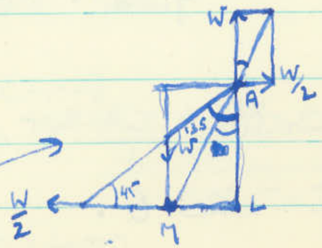


moms. A $W - 2Y + 2X = 0$

C $3W - 2Y - 2X = 0$.

Add, subt.

$Y = W \quad X = \frac{W}{2}$.



$\frac{AL}{LM} = \frac{2}{1}$

$\frac{AL}{BL} = \frac{1}{1}$

D of lect 7. p.69 no. 3.

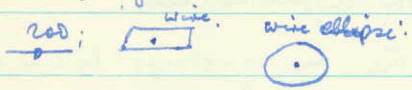
a check on the result "Lami's Theorem"

Maths Phy. lect IIx Nov 2nd '59.

A. (i) C. of G. formula: $\bar{x} = \frac{\Sigma w_i x_i}{\Sigma w_i}$ [explain "moments" idea]

(ii) ~~rod~~, symmetrical bodies have C. of G. in C. of sym.

1 dim rod; wire. wire ellipse:

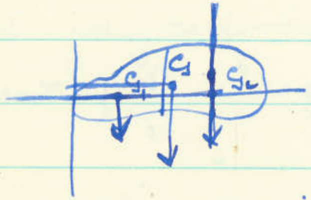


2 dim

3 dim

[tetrahedron, cone: divides line from vertex to C. of base in ratio 3:1.

(ii)



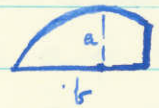
$W_1 x_1 + W_2 x_2 = \bar{x}(W_1 + W_2)$

given $\overset{W_1}{x_1}, \overset{W_2}{x_2} \rightarrow \bar{x}, W$.

given $\overset{W_1}{x_1}, \overset{W}{\bar{x}} \rightarrow x_2, W - W_1$.

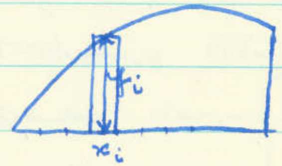
Start at B: return to A.

B. C. of G. of uniform plate:



divide in N strips; width $\frac{b}{n}$

x_i, y_i C. of G. of strip.



$$\bar{x} = \frac{\sum_{1 \ldots N} w_i x_i}{\Sigma w_i}$$

if w_0 weight per unit area then $w_i = w_0 y_i \frac{b}{n}$

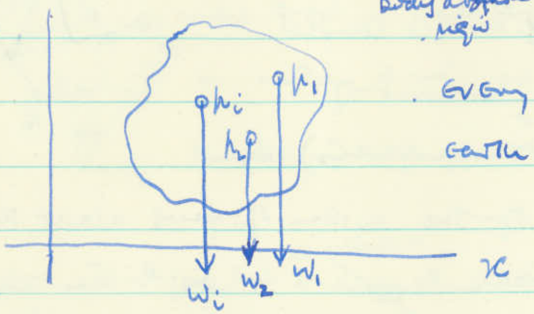
$$= \frac{\sum_{i=1 \ldots N} y_i \cdot x_i \cdot \frac{b}{n}}{\sum_{i=1 \ldots N} y_i \cdot \frac{b}{n}}$$

Maths Phy. Lect VII Oct 30th 59.

A. **Centre of Gravity:**

We are considering a particular force here, namely, the attraction the earth has for a body. We assume ~~that the body is small, so that the forces are parallel.~~

Consider. Body a system of N "particles" ~~body~~



. every particle p_i attracted to earth by force w_i.

we assume the body is small enough to consider all such forces to be parallel.

Now these parallel forces have as we know a resultant, parallel to themselves, equal to Σw_i, which we can get by compounding the forces in pairs...

take p_1 p_2. ~~now resultant~~ if the force w_1 w_2 on them intersect the x axis at x_1 x_2 then resultant intersects x axis at

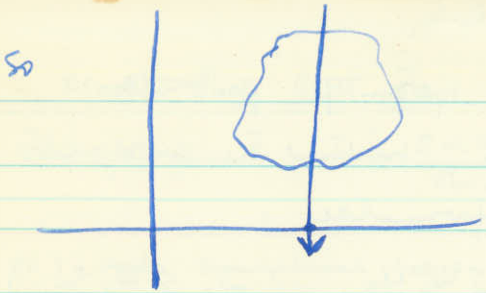
$$x_{12} = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2} \quad \text{(coord. geom.)}$$

Compound this with w_3:

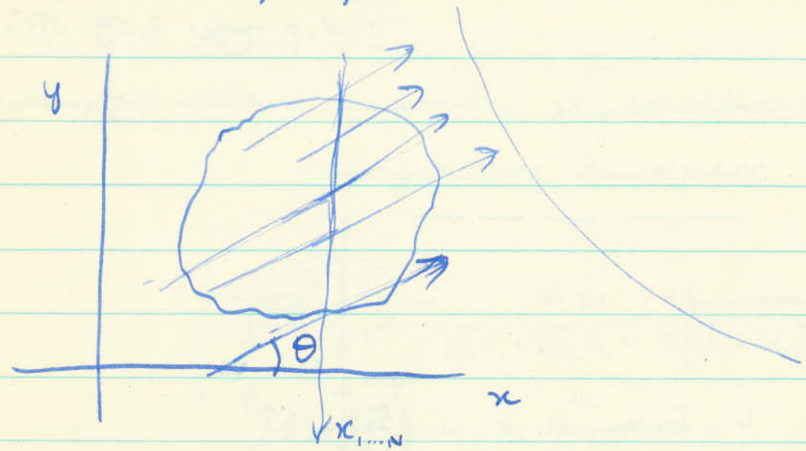
$$x_{123} = \frac{(w_1+w_2)\frac{w_1 x_1 + w_2 x_2}{w_1 + w_2} + w_3 x_3}{w_1 + w_2 + w_3}$$

Eventually $x_{1\ldots N} = \frac{\Sigma w_i x_i}{\Sigma w_i}$

So



Now we go further:



resolve forces: $\parallel$ to 2 axes: $w_i \cos\theta$; $w_i \sin\theta$ typical.

result. in "vert." direction along the line

$$x_{1\ldots N} = \frac{\Sigma w_i \cos\theta . x_i}{\Sigma w_i \cos\theta} = \frac{\Sigma w_i x_i}{\Sigma w_i}$$

res. in horizontal dir. acts along the line

$$y_{1\ldots N} = \frac{\Sigma w_i \sin\theta . x_i}{\Sigma w_i \sin\theta} = \frac{\Sigma w_i y_i}{\Sigma w_i}$$

These intersect at pt $\left(\frac{\Sigma w_i x_i}{\Sigma w_i}, \frac{\Sigma w_i y_i}{\Sigma w_i}\right)$ a fixed pt. for all θ.

[This means that "moving the earth round" etc: ∴].

We call the fixed pt. the C. of G.

Maths Phy. Lect VI Oct 28th.

A. Friction: general discussion.

1) against motion 2) keeps body in equil. but limit

limiting friction 3) $= \mu R$

4) indep. of ext. or shape provided R unaltered.

(5) in motion $< \mu R$. op. mot.)

So $\frac{F^{lim}}{R} = \mu$.

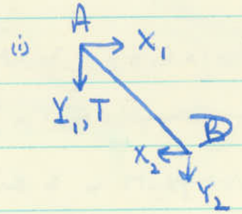
$\tan\lambda = \mu$ λ "angle of fr" "μ" coeff of friction".

B. Friction etc at joint "ball + socket"

(i) if no friction : R "someway" e.g.

(ii) if friction, etc "unknown couple"

C. Problem in Loney: take part of one



(ii) work out W and add in.

(iii) use method 2: reduce to force at arb. pt, + couple.

3 eqn $X_1 - X_2 = 0$

$W + Y_1 + Y_2 + T = 0$

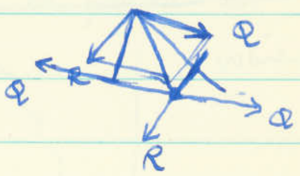
$Wa\cos\theta + X_2 \cdot 2a\sin\theta + Y_2 \cdot 2a\cos\theta = 0.$

(iv) use method 1.

By method 2 we have already reduced the forces to a single force and a single couple, the pt. A being arbitrarily chosen.

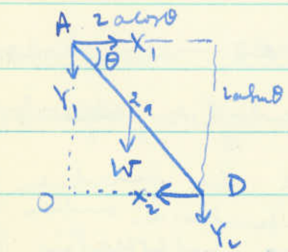
Now can reduce further (as before).

so a) single force OR b) single couple.



We can get the moment of this single force about a pt. (or the m. of the C.) by taking moments of the system of forces about that pt.

If the pt. choosen happens to lie on R then we get zero: but we get zero for 3 non-col. pts iff $\begin{Bmatrix} R=0 & G=0 \end{Bmatrix}$ system in equil.



moments about

A: $Wa\cos\theta + X_2 2a\sin\theta + Y_2 2a\cos\theta = 0$

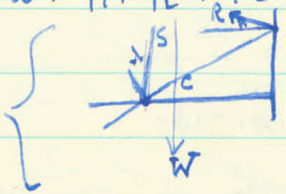
D: $Wa\cos\theta - X_1 2a\sin\theta + Y_1 2a\cos\theta + T \cdot 2a\cos\theta = 0.$

O: $Wa\cos\theta + X_1 2a\sin\theta + Y_2 2a\cos\theta = 0$

$A - O \rightarrow X_2 - X_1 = 0.$

$D + O \rightarrow W + Y_1 + Y_2 + T = 0.$

D. Loney pg. 69. no. 3: Begin it.



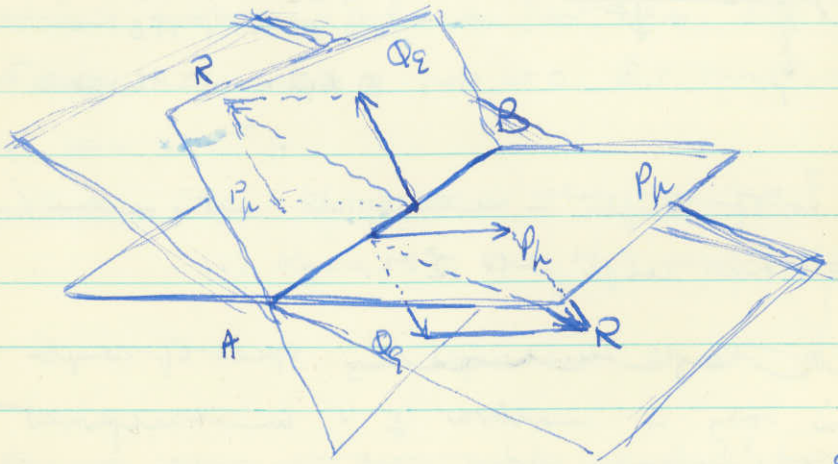
E. mention whats coming: e.g. C. of G. (3) V.W (3) Bending (3) Eq. of strings (5)

4. So a couple may be replaced by any other couple acting in a $\parallel^{l}$ plane provided that the moments of the 2 couples are equal.

(i) mom.
(ii) direct.
(iii) sense. antic. posit.

4^a 3 forces may dir. line ≡ couple $M = 2\Delta$.

5



result. in couple moment R, acting in a plane thro' AB. ~~dir~~ couple got by $\square^{la}$ to Pp, Qq.
~~directio~~ swing $\parallel^{l}$ round R. Then Qq axis, Pp axis, R axis: so mag. & dir. (& sense) of rectangle got by compounding axis ala $\square$ law.

Maths Phy. Lect 5. 26th Oct.

Equilibrium conditions.

A. Particle: a) polygon of forces; b) triangle or Lami; c) resolve.

B. Rigid Body (i) if 3 forces:

They must meet in a pt: → Because: The resultant of 2 of them must have the same line of action as the 3rd.

∴ use ~~conditions~~ 1) meet in a pt. 2) Lami's Theorem.

(ii) ~~if~~ any no. of forces:

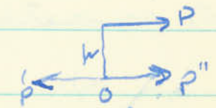
a) ≡ single force or single couple: → couples + forces ↓ (compound $\parallel^{l}$ forces.) couple^A + non-$\parallel^{l}$ forces^B ↓ couple A + couple^B + non-$\parallel^{l}$ f ↓ couple + non-$\parallel^{l}$ f. if one zero: ~~a~~ proved; if 2 non-zero: compound → single fo

∴ moments about 3 non-col. pts = 0 } 3 eqn.

b) ≡ a force (= R(A, B, C, ..) at arb. pt O, + a couple (Σ Pp)

∴ resolved parts in 2 dir. zero and moment about arb. pt. zero. } 3 eqn.

take any force P



P ≡ P at O + couple Pp.

C. Rules: 1. Draw < realistic figure / workable sketch.
2. Mark forces (including reactions, weights, tensions)
3. Get 3 eqn. for each body.
4. Write down any geometrical relations.

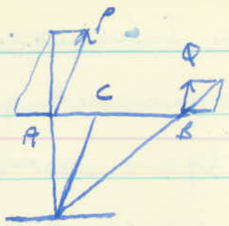
E. Eqn. of Resultant

D. Remarks re 2. (i) "tension" (ii) reaction 3. use B. 4. "projections"

F. give 2 problems
Loney p49 {No. 11, No. 14

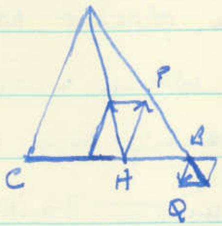
Maths Phys. lect. 3. 21st Oct 59.

A. Parallel forces a) like



P.AC = Q.BC.

b) unlike



P.AC = Q.BC.

B. A^t fails if $P = -Q$ in mag. & dir.

∴ no single force equivalent to P and Q.

To deal with this case we introduce a new notion: we get it by examining our experience of forces on fixed bodies: diag: →F : F can't move body as a whole if fixed at O: but "turns": measure of turning power is F.p.

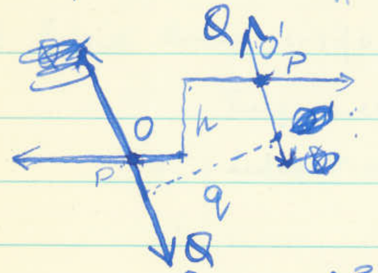
2 Theorems: (i) M = 2 area Δ whose base reps F with vertex O.

(ii) Algebraic Σ of Ms = M. of resultant — O outside, O inside, parallel {like, unlike}

C. Now, we return to our unlike equal ∥ forces: we expect them to have some relation to rotation: we examine their "turning power" & find that it is constant about any pt. in the plane. and = F.p. We call the pair of forces a couple and the F.P ~~turning power~~ the moment of the couple.

Maths Phys. lect 4.

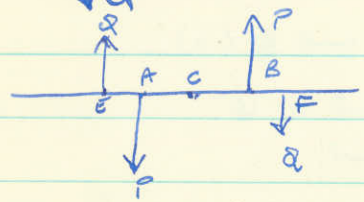
A. Next we want to show that 2 couples acting in one plane (on a rigid body) whose moments are equal and opposite balance one another.



R(PQ) one pass thro O : dir. OO'

R(PQ)' pass thro O'. O'O

equal in mag. since Δ Δ same ∴.



Result. PQ↑ acts thro C

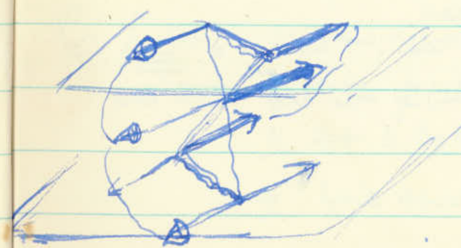
Q.EC = P.BC

Q.EF = P.AB

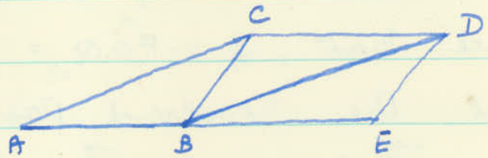
∴ Q.EF = P.AC. cond. that R thro'

B. Any no. of couples in the same plane acting on a rigid body ≡ single couple, M = ΣM.

C. Effect of couple on a rigid body is unaltered if it be transferred to any plane ∥ to its own, the arm remaining ∥ to original pos.



(i) if CA represents Z then the forces are in equilibr.
(ii) if the forces are in equilibrium that CA reps. Z.



Continue AB to E so that BE = AB. Then we know (by completing the parallelogram) that BD reps. the Resultant, R, of X and Y.

(i) Suppose that CA happens to represent Z. Now since ABDC is a parallelogram AC is equal and parallel to BD and hence represents R in magnitude and direction. Hence $R = -Z$ and the forces are in equilibrium.

(ii) Suppose the forces are in equilibrium. We want to show that CA reps. Z. Now we know that AC represents the resultant R of X and Y. But if the forces are in equilibrium then Z must be equal, but opposite to, the resultant. Therefore Z must be represented by CA.

2nd lecture: Summary.

(i) "If 2 forces acting at a pt. be rep. in mag. + dir. by the sides of a parallelogram drawn from one of its angular pts, their resultant is rep. both in mag. + dir. by the diag. of the parallelogram passing thro' that angular pt.
get $R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$ and $\tan\theta = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$.

(ii) Resol. of forces: (a) any 2 direction

(b) $\perp$.

"resolved parts" $F\cos\theta$" $F\sin\theta$

Consequences of (i)

(iii) Δ of forces: if 3 forces, acting at a pt, be rep. in mag. + dir. by the sides of a Δ taken in order then they are in eq.
(conversely if lines ∥ to 3 forces, Δ form reps. forces in mag. + dir.)

(iv) Lami's Th: $P \sim \sin(QR)$.

(v) ⬠ of forces: if any no of forces acting at a pt be rep in mag. + dir. by the sides of a ⬠ taken in order they are in eql.
(no converse).

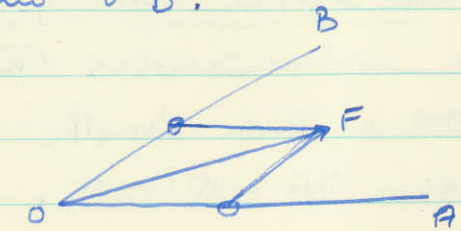
(vi) The resultant of any no of forces acting at a pt: resolve in 2 (3) $\perp$ dirs. $\Sigma x_i = X$ $R = X^2 + Y^2 + Z^2 = 0$ if $x = 0$ etc.

(vii) Examples: a)

b)

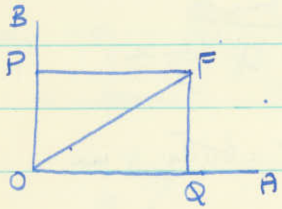
$\frac{\mu}{AB}\sin ABC - \frac{\mu}{BC}\sin BAC = 0$.

(viii) ∥ᶜ forces: a) like b) unlike (one exception) any given force (represented by OF, say) in any two arbitrary directions (OA, OB say) merely by drawing parallels thro' F to OA and OB:



The points of intersection of the parallels on the lines OA and OB give the magnitudes of the required components.

The most important case occurs when the angle BOA is a right angle: in this



special case the components are called the "resolved parts" of the original force. We see immediately that the resolved part OQ $= OF \cos F\hat{O}Q$. And that $OP = OF \cos F\hat{O}P$.

In general, the Resolved Part of a given force in a given direction is

obtained by multiplying the given force by the cosine of the angle between the force and the given direction.

Notice that, if $F\hat{O}Q = \alpha$ then $F\hat{O}P = \frac{\pi}{2} - \alpha$

So that the resolved parts are

$\vec{OQ} = \vec{OF} \cos\alpha \qquad \vec{OP} = \vec{OF}\cos(\frac{\pi}{2}-\alpha) = \vec{OF}\sin\alpha.$

A few immediate consequences of the Parallelogram principle are a) The principle of the Triangle of Forces b) Lami's Theorem c) The principle of the polygon of forces.

a) A necessary and sufficient condition that three forces acting at a point be represented in magnitude and direction by the sides of a triangle is that they be in equilibrium.

Let X, Y, Z be three forces acting at a pt. Let AB represent X, BC represent Y. Join CA. Then we want to prove that

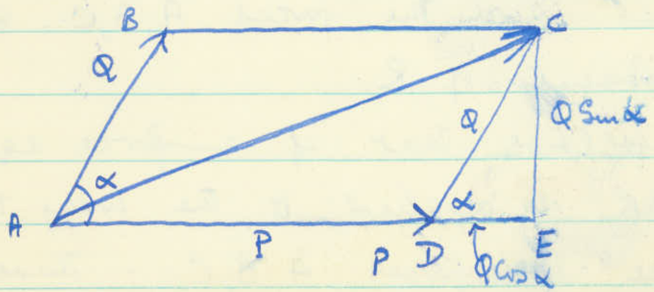
At this stage we introduce the fundamental ~~theorem~~ principle of Statics, a principle based on experimental evidence:

The Parallelogram Principle:

If two forces, acting at a point, be represented in magnitude and direction by the two sides of a parallelogram drawn from one of its angular points, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram passing through that angular point.

(end of 1st lect.). 2nd lect.

From this principle it follows that the calculation of the resultant R, of any two forces acting at a point is a matter of geometry:



Let the lines AD and AB represent in magnitude and direction the forces P and Q acting at an angle α. Then $B\hat{A}D = \alpha$. Complete the parallelogram, joining AC, and dropping a perpendicular CE on AD. Then since CD = AB = Q in length,

Then $CE = Q \sin \alpha$, $DE = Q \cos \alpha$.

~~and if R is the~~ The magnitude of the Resultant R is represented by AC.

$$\therefore \quad R = \sqrt{(P + Q\cos\alpha)^2 + Q^2 \sin^2\alpha}$$
$$= \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

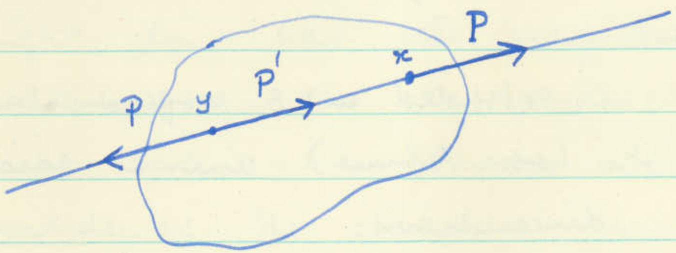
We can also get the direction in which it acts, for

$$\tan CAD = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

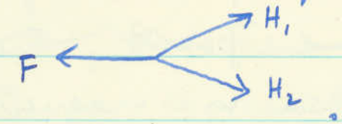
So we have the resultant both in magnitude and direction.

The above process is one of "compounding" forces, but by the same parallelogram principle we can "resolve" ("break up")

The original force will balance the force equal and opposite to it at the 2^{nd} pt., thus leaving a force P acting at the 2^{nd} pt.



We saw that a force has magnitude and direction (i.e. is a vector), and can be represented by a straight line. If the forces to be considered all lie in one plane, then they can be represented by lines in a plane. For example, the forces involved in 2 horses pulling a barge against the flow ~~current~~ might be represented thus:



Def:

If two or more forces, A, B, C... act upon a rigid body and if a single force, R, can be found whose effect upon the body is the same as that of the forces A, B, C... This single force R is called the Resultant of the other forces and the forces A, B, C... are called the components of R.

It follows that if a force equal and opposite to R be applied to the body that it will balance the forces A, B, C... : there will be equilibrium.

Topp?

[From experience we know that a force has both magnitude and direction. Anything in physics which has both magnitude and direction is called a vector, and since a straight line has both magnitude and direction we can represent vectors by straight lines. We choose a force of a certain magnitude as the standard unit (in our case, the weight of one gram mass) and measure the other forces by comparison with it as unit.]

We must also consider the types of bodies on which forces act: there are four general types:

i) a particle, considered to have no extension, (nor surface area, nor volume) and is said to be of zero dimensions.

ii) a thin rod (i.e. practically a line) has extension in one direction (but still no surface area or volume): it is said to be

of one dimension.

iii) a lamina (e.g. a thin plate of glass) has extension in 2 directions (has area but no volume) and therefore is said to be of two dimensions.

iv) a Body is extended in 3 perpendicular directions (it has volume) and is said to be of three dimensions.

A body (lamina, rod) is rigid if the distance between any 2 arbitrary points remains constant: if this is not true it is "deformable" (elastic or fluid). Most of our considerations are restricted to rigid bodies. When a force acts on a rigid body at any particular point it may be considered to act at any other point in the line along which it acts: for, we can introduce a pair of equal and opposite forces, equal and ‖el to the given force P, at any point on the line of action, and we assume that

a) A force of attraction: when a force is exerted without visible medium e.g. the attraction which the earth has for any body. This attraction is equal to $\frac{GM_E M_B}{r^2}$ where G is a constant, M_E and M_B are constants associated with the earth and the body respectively, and r is the distance between their centres (of gravity: to be defined later). Writing g for $\frac{GM_E}{r^2}$ when the body is on the surface of the earth (so that r is then the radius of the earth), the force is equal to gM_B. This force is called the weight of the body, W_B, so that $W_B = gM_B$. The significance of the constant g will appear more clearly in dynamics. The constant M_B is called the mass of the body.

b) A force of reaction: we assume that everything material that acts suffers an equal and opposite reaction e.g. when you press down on the chair the chair presses you up. Such forces of reaction must be taken account

of in our considerations.

c) The force exerted by means of a string is called a tension, and it acts along the string. If the string stretches under tension it is called elastic and we assume (with experimental backing) that the tension is related to the change in length by the formula $T = \lambda \frac{x-a}{a}$, where T is the tension, x and a the extended and the original length respectively, and λ a constant for the particular string called the "modulus of elasticity".

d) The force between two rough bodies when they slide, or tend to slide, over one another is called the force of friction: we will consider it later.

The first year mathematical physics course consists mainly of classical mechanics in two dimensions. Mechanics is the science of motion and falls naturally into 2 main sections, according as things are moving or not moving. Dynamics deals with moving bodies, Statics deals with bodies at rest.

We begin the course with Statics, treating it as a science independent of dynamics, and we consider immediately some of its basic notions and assumptions, all of which have their foundation in our ordinary experience:

We define a force in statics as anything which tends to change the state of rest. When two or more forces act upon a body and are so arranged that the body remains at rest, then the forces are said to be in equilibrium. Two forces are equal if they balance one another when acting in opposite directions. put in "vector" section here. Some of the types of forces which occur are